Consider the group $\mathcal{S}_4$ of permutations on the four-element set $X = \{1,2,3,4\}$ under the operation of function composition. Let A_4 be the subcollection

$$E = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \quad K = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

$$N = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \quad O = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

The set A_4 forms a subgroup of the group $\mathcal{S}_4$; its operation table is provided below. (Remember, the operation is function composition.)

$\circ$	E	G	H	I	J	K	L	M	N	O	P	Q
E	E	G	H	I	J	K	L	M	N	O	P	Q
G	G	H	E	P	Q	O	J	K	I	M	N	L
H	H	E	G	N	L	M	Q	O	P	K	I	J
I	I	J	K	E	G	H	O	P	Q	L	M	N
J	J	K	I	M	N	L	G	H	E	P	Q	O
K	K	I	J	Q	O	P	N	L	M	H	E	G
L	L	M	N	O	P	Q	E	G	H	I	J	K
M	M	N	L	J	K	I	P	Q	O	G	H	E
N	N	L	M	H	E	G	K	I	J	Q	O	P
O	O	P	Q	L	M	N	I	J	K	E	G	H
P	P	Q	O	G	H	E	M	N	L	J	K	I
Q	Q	O	P	K	I	J	H	E	G	N	L	M

Let $\boldsymbol{\mathcal{A}}_4 = (A_4, \circ)$ be the group whose operation table is shown above, and consider the function $\varphi : A_4 \to \mathbb{Z}_3$ defined by the following rule:

$$\varphi(E) = 0 \quad \varphi(G) = 2 \quad \varphi(H) = 1 \quad \varphi(I) = 0 \quad \varphi(J) = 2 \quad \varphi(K) = 1$$

$$\varphi(L) = 0 \quad \varphi(M) = 2 \quad \varphi(N) = 1 \quad \varphi(O) = 0 \quad \varphi(P) = 2 \quad \varphi(Q) = 1$$

This function is a group homomorphism from $\boldsymbol{\mathcal{A}}_4$ to $\boldsymbol{\mathcal{Z}}_3$. (You may assume this.)

Problem 1. If $f : X \to Y$ is any function and $v \in Y$, then we define the *preimage* of v under f to be the set $\text{Pre}_f(v) = \{u \in X : f(u) = v\}$. (Compare to Homework Problem 4 of Investigation 10.) What is $\text{Pre}_\varphi(v)$ for each $v \in \mathbb{Z}_3$?

Problem 2. In the table below, the elements of A_4 have been sorted by preimage under the function φ.

	$\text{Pre}_\varphi(0)$				$\text{Pre}_\varphi(1)$				$\text{Pre}_\varphi(2)$			
$\circ$	E	I	L	O	H	K	N	Q	P	M	J	G
E												
I												
L												
O												
H												
K												
N												
Q												
P												
M												
J												
G												

Part (a). Fill in this rearranged table.

Part (b). What are some patterns you notice in the rearranged table?

Problem 3. Based on the rearranged table above, fill in the table below so that

a) The table defines $\otimes$ as a binary operation on the set $P_\varphi = \{\text{Pre}_\varphi(0), \text{Pre}_\varphi(1), \text{Pre}_\varphi(2)\}$.
b) The algebra $(P_\varphi, \otimes)$ is a group.

$\otimes$	$\text{Pre}_\varphi(0)$	$\text{Pre}_\varphi(1)$	$\text{Pre}_\varphi(2)$
$\text{Pre}_\varphi(0)$			
$\text{Pre}_\varphi(1)$			
$\text{Pre}_\varphi(2)$			

To what group is the algebra $(P_\varphi, \otimes)$ isomorphic? Justify your answer.

Let $S_\perp = \{RR, R, F, RF\}$ and let $\boldsymbol{\mathcal{S}}_\perp = (S_\perp, *)$ be the rectangle symmetries group (See Homework Problem 8 of Investigation 8), and consider the group $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{S}}_\perp$.

$\uplus$	$(0,RR)$	$(1,RR)$	$(2,RR)$	$(0,R)$	$(1,R)$	$(2,R)$	$(0,F)$	$(1,F)$	$(2,F)$	$(0,RF)$	$(1,RF)$	$(2,RF)$
$(0,RR)$	(0,RR)	(1,RR)	(2,RR)	(0,R)	(1,R)	(2,R)	(0,F)	(1,F)	(2,F)	(0,RF)	(1,RF)	(2,RF)
$(1,RR)$	(1,RR)	(2,RR)	(0,RR)	(1,R)	(2,R)	(0,R)	(1,F)	(2,F)	(0,F)	(1,RF)	(2,RF)	(0,RF)
$(2,RR)$	(2,RR)	(0,RR)	(1,RR)	(2,R)	(0,R)	(1,R)	(2,F)	(0,F)	(1,F)	(2,RF)	(0,RF)	(1,RF)
$(0,R)$	(0,R)	(1,R)	(2,R)	(0,RR)	(1,RR)	(2,RR)	(0,RF)	(1,RF)	(2,RF)	(0,F)	(1,F)	(2,F)
$(1,R)$	(1,R)	(2,R)	(0,R)	(1,RR)	(2,RR)	(0,RR)	(1,RF)	(2,RF)	(0,RF)	(1,F)	(2,F)	(0,F)
$(2,R)$	(2,R)	(0,R)	(1,R)	(2,RR)	(0,RR)	(1,RR)	(2,RF)	(0,RF)	(1,RF)	(2,F)	(0,F)	(1,F)
$(0,F)$	(0,F)	(1,F)	(2,F)	(0,RF)	(1,RF)	(2,RF)	(0,RR)	(1,RR)	(2,RR)	(0,R)	(1,R)	(2,R)
$(1,F)$	(1,F)	(2,F)	(0,F)	(1,RF)	(2,RF)	(0,RF)	(1,RR)	(2,RR)	(0,RR)	(1,R)	(2,R)	(0,R)
$(2,F)$	(2,F)	(0,F)	(1,F)	(2,RF)	(0,RF)	(1,RF)	(2,RR)	(0,RR)	(1,RR)	(2,R)	(0,R)	(1,R)
$(0,RF)$	(0,RF)	(1,RF)	(2,RF)	(0,F)	(1,F)	(2,F)	(0,R)	(1,R)	(2,R)	(0,RR)	(1,RR)	(2,RR)
$(1,RF)$	(1,RF)	(2,RF)	(0,RF)	(1,F)	(2,F)	(0,F)	(1,R)	(2,R)	(0,R)	(1,RR)	(2,RR)	(0,RR)
$(2,RF)$	(2,RF)	(0,RF)	(1,RF)	(2,F)	(0,F)	(1,F)	(2,F)	(0,R)	(1,R)	(2,RR)	(0,RR)	(1,RR)

Problem 4. Consider the function $\vartheta : \mathbb{Z}_3 \times S_\perp \to \mathbb{Z}_4 \times \mathbb{Z}_8$ defined by the following rule:

$$\vartheta\big((0,RR)\big) = (0,0) \quad \vartheta\big((0,F)\big) = (2,0) \quad \vartheta\big((0,R)\big) = (2,4)$$

$$\vartheta\big((1,RR)\big) = (0,0) \quad \vartheta\big((1,F)\big) = (2,0) \quad \vartheta\big((1,R)\big) = (2,4)$$

$$\vartheta\big((2,RR)\big) = (0,0) \quad \vartheta\big((2,F)\big) = (2,0) \quad \vartheta\big((0,RF)\big) = (0,4)$$

$$\vartheta\big((1,RF)\big) = (0,4) \quad \vartheta\big((2,R)\big) = (2,4) \quad \vartheta\big((2,RF)\big) = (0,4)$$

This function is a group homomorphism from $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{S}}_\perp$ to $\boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_8$. (You may assume this.)

Part (a). Are there any members of $\mathbb{Z}_4 \times \mathbb{Z}_8$ that have an empty preimage under the function ϑ?

Part (b). What is $\mathrm{Pre}_\vartheta(v)$ for each $v \in \mathbb{Z}_4 \times \mathbb{Z}_8$ that has a nonempty preimage?

Part (c). In light of Problem 7 from Investigation 10, we know that $\vartheta(\mathbb{Z}_3 \times S_\perp)$ is a subgroup of $\boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_8$. Write down the operation table for this subgroup.

Problem 5. In the table below, the elements of $\mathbb{Z}_3 \times S_\perp$ have been sorted by preimage under the function ϑ.

	$\mathrm{Pre}_\varphi((0,0))$			$\mathrm{Pre}_\varphi((2,0))$			$\mathrm{Pre}_\varphi((2,4))$			$\mathrm{Pre}_\varphi((0,4))$		
⊍	(0,RR)	(1,RR)	(2,RR)	(0,F)	(1,F)	(2,F)	(0,R)	(1,R)	(2,R)	(0,RF)	(1,RF)	(2,RF)
(0,RR)												
(1,RR)												
(2,RR)												
(0,F)												
(1,F)												
(2,F)												
(0,R)												
(1,R)												
(2,R)												
(0,RF)												
(1,RF)												
(2,RF)												

Part (a). Fill in this rearranged table.

Part (b). What are some patterns you notice in the rearranged table?

Problem 6. Based on the rearranged table above, fill in the table below so that

a) The table defines $\otimes$ as a binary operation on the set
$$P_\vartheta = \{\mathrm{Pre}_\vartheta((0,0)), \mathrm{Pre}_\vartheta((2,0)), \mathrm{Pre}_\vartheta((2,4)), \mathrm{Pre}_\vartheta((0,4))\}.$$
b) The algebra $(P_\vartheta, \otimes)$ is a group.

$\otimes$	$\mathrm{Pre}_\vartheta((0,0))$	$\mathrm{Pre}_\vartheta((2,0))$	$\mathrm{Pre}_\vartheta((2,4))$	$\mathrm{Pre}_\vartheta((0,4))$
$\mathrm{Pre}_\vartheta((0,0))$				
$\mathrm{Pre}_\vartheta((2,0))$				
$\mathrm{Pre}_\vartheta((2,4))$				
$\mathrm{Pre}_\vartheta((0,4))$				

Is the algebra $(P_\vartheta, \otimes)$ isomorphic to the subgroup $\vartheta(\mathbb{Z}_3 \times S_\perp)$? Justify your answer.

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $\boldsymbol{\mathcal{Y}} = (Y,\diamond)$ be another group such that $f : X \to Y$ is a group homomorphism from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}$. Based on the results from the previous problems, it seems that we can sort the elements of X by preimage and create a "preimage algebra" on these sorted sets that is isomorphic to the subgroup $f(X)$ of the group $\boldsymbol{\mathcal{Y}}$. Let's prove that this is the case.

Problem 7. Consider the preimage algebra $(P_\varphi,\otimes)$ you created in Problem 3. Using your table, verify that $\mathrm{Pre}_\varphi(v) \otimes \mathrm{Pre}_\varphi(w) = \mathrm{Pre}_\varphi(v \boxplus_3 w)$.

Problem 8. Consider the preimage algebra $(P_\vartheta,\otimes)$ you created in Problem 6. In this case, is it still true that $\mathrm{Pre}_\vartheta(v) \otimes \mathrm{Pre}_\vartheta(w) = \mathrm{Pre}_\vartheta(v \uplus w)$?

Theorem 11.1 Preimage Groups

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $\boldsymbol{\mathcal{Y}} = (Y,\diamond)$ be another group such that $f : X \to Y$ is a group homomorphism from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}$. Let $P_f = \{\mathrm{Pre}_f(v) : v \in f(X)\}$. If we define a binary operation $\otimes$ on P_f according to the rule

$$\mathrm{Pre}_f(v) \otimes \mathrm{Pre}_f(w) = \mathrm{Pre}_f(v \diamond w)$$

then the algebra $(P_f,\otimes)$ is a group that is isomorphic to the subgroup $f(X)$ of the group $\boldsymbol{\mathcal{Y}}$.

In the following exercises, we will prove Theorem 11.1.

Problem 9. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $\boldsymbol{\mathcal{Y}} = (Y,\diamond)$ be another group such that $f : X \to Y$ is a group homomorphism from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}$.

Part (a). Explain why we know that $\otimes$ as defined above is truly a binary operation on the set P_f.

Part (b). Prove that the operation $\otimes$ as defined above is associative.

Problem 10. Complete the proof that the algebra $(P_f,\otimes)$ is a group.

Problem 11. Consider the function $g_f : P_f \to f(X)$ defined by $g_f\left(\mathrm{Pre}_f(v)\right) = v$. Prove that the function g_f is an isomorphism.

Homework.

Problem 1. Let $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\diamond)$ be groups and consider the product group $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}$. In Problem 1 of Investigation 10, you showed that $\pi_X : X \times Y \to X$ be defined by $\pi_X[(a,b)] = a$ is a group epimorphism (a surjective group homomorphism).

Part (a). Theorem 11.1 therefore tells us that $(P_{\pi_X},\otimes)$ is isomorphic to the group $\boldsymbol{\mathcal{X}}$. For each $a \in X$, explain why $\mathrm{Pre}_{\pi_X}(a) = \{(a,u) : u \in Y\}$.

Part (b). Is it true that we could define $\otimes$ by the rule $\mathrm{Pre}_{\pi_X}(a) \otimes \mathrm{Pre}_{\pi_X}(b) = \{(a * b, v) : v \in Y\}$? Explain.

Problem 2. Consider the product group $\boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_2$ and the cross symemtries group $\mathcal{S}_\times = (S_\times,*)$, along with the function $\gamma : \mathbb{Z}_4 \times \mathbb{Z}_2 \to S_\times$ defined by the following rule.

$$\gamma\big((0,0)\big) = RRRR \quad \gamma\big((0,1)\big) = FR \quad \gamma\big((1,0)\big) = (FR)(RR) \quad \gamma\big((1,1)\big) = RR$$

$$\gamma\big((3,0)\big) = (FR)(RR) \quad \gamma\big((3,1)\big) = RR \quad \gamma\big((2,1)\big) = FR \quad \gamma\big((2,0)\big) = RRRR$$

Part (a). Prove that γ is a group homomorphism from $\boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_2$ to $\mathcal{S}_\times$.

Part (b). We know that $\gamma(\mathbb{Z}_4 \times \mathbb{Z}_2) = \{RRRR, FR, (FR)(RR), RR\}$ is a subgroup of $\mathcal{S}_\times$. To what familiar group is this subgroup isomorphic? Justify your answer.

Part (c). Construct the members of P_γ.

Part (d). Construct the operation table for the preimage group $(P_\gamma,\otimes)$ and verify that it is isomorphic to the subgroup $\gamma(\mathbb{Z}_4 \times \mathbb{Z}_2)$.

Problem 3. Consider the homomorphisms φ and ϑ from this investigation. What are the members of $\ker(\varphi)$ and $\ker(\vartheta)$? What role do these sets play in the preimage groups?

Problem 4. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $\boldsymbol{\mathcal{Y}} = (Y,\diamond)$ be another group such that $f : X \to Y$ is a group homomorphism from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}$. Let $v \in f(X)$, and suppose $a \in \text{Pre}_f(v)$.

Part (a). Prove that $x \in \text{Pre}_f(v)$ if and only if $a^{-1} * x \in \ker(f)$.

Part (b). Prove that $\text{Pre}_f(v) = \{a * u : u \in \ker(f)\}$.

Problem 5. Let n be a fixed positive integer and consider the function $R : \mathbb{Z} \to \mathbb{Z}_n$ defined by the formula $R(a) = r$, where r is the remainder for a relative to n. In Homework Problem 3 of Investigation 10, you proved that this function is a group epimorphism from $\boldsymbol{\mathcal{Z}}$ to $\boldsymbol{\mathcal{Z}}_n$.

Part (a). Theorem 11.1 tells us that the preimage group $(P_R,\otimes)$ is isomorphic to $\boldsymbol{\mathcal{Z}}_n$, and Problem 3 tells us that $\ker(R)$ serves as the identity for $(P_R,\otimes)$. What are the members of $\ker(R)$?

Part (b). Explain why $P_R = \big\{\{r + u : u \in n\mathbb{Z}\} : r \text{ is an integer and } 0 \le r < n\big\}$.

Problem 6. Let $\mathcal{R}' = (\mathbb{R}',\cdot)$ represent the group of nonzero real numbers under multiplication. Consider the General Linear Group $\boldsymbol{GL_2} = (U_{2\times2},*)$ of 2×2 invertible matrices with real number entries under matrix multiplication, along with the function $f : U_{2\times2} \to \mathbb{R}'$ defined by $f(A) = \text{Det}(A)$.

Part (a). Use the properties of determinants to prove that f is a group homomorphism from $\boldsymbol{GL_2}$ to $\mathcal{R}'$.

Part (b). Prove that f is a surjection.

Part (c). Theorem 11.1 tells us that the preimage group $(P_f,\otimes)$ is isomorphic to $\mathcal{R}'$, and Problem 3 tells us that $\ker(f)$ serves as the identity for $(P_f,\otimes)$. What are the members of $\ker(f)$?

Part (d). Let α be a nonzero real number and prove that $\text{Pre}_f(\alpha) = \{M_\alpha * U : U \in \ker(f)\}$, where

$$M_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

Part (e). Find another matrix N such that $\text{Pre}_f(\alpha) = \{N * V : V \in \ker(f)\}$. What is your strategy for finding this matrix?