In this investigation, we will explore a question that naturally arises from the previous investigation --- Is *every* subgroup of a group necessarily the kernel of *some* group homomorphism; and, as such, is every subgroup the identity of *some* preimage group?

In order to answer this question, we will need to find a way of describing preimage groups that does not involve group homomorphisms.

Cosets in a Group

Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let H be any subgroup of $\boldsymbol{\mathcal{X}}$. For each $a \in X$, the *left coset* of H generated by a is defined to be the set

$$a\,H = \{a * y : y \in H\}$$

We will let $LC_H = \{a\,H : a \in X\}$.

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $\boldsymbol{\mathcal{Y}} = (Y,\diamond)$ be another group such that $f : X \to Y$ is a group homomorphism from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}$. Notice that in Homework Problem 4 of Investigation 11, you showed that for all $v \in f(X)$, we have $\mathrm{Pre}_f(v) = a\,\ker(f)$ for *every* $a \in \mathrm{Pre}_f(v)$. Hence, the elements of the preimage group are simply the left cosets of the kernel.

Recall the cross-symmetries group $\mathcal{S}_\times = (S_\times,*)$ introduced in Investigation 3. You identified the ten subgroups of this group in Investigation 9.

Here is the operation table for the cross symmetries group.

$*$	$RRRR$	RRR	RR	R	F	FR	$F(RR)$	$(FR)(RR)$
$RRRR$	$RRRR$	RRR	RR	R	F	FR	$F(RR)$	$(FR)(RR)$
RRR	RRR	RR	R	$RRRR$	FR	$F(RR)$	$(FR)(RR)$	F
RR	RR	R	$RRRR$	RRR	$F(RR)$	$(FR)(RR)$	F	FR
R	R	$RRRR$	RRR	RR	$(FR)(RR)$	F	FR	$F(RR)$
F	F	$(FR)(RR)$	$F(RR)$	FR	$RRRR$	R	RR	RRR
FR	FR	F	$(FR)(RR)$	$F(RR)$	RRR	$RRRR$	R	RR
$F(RR)$	$F(RR)$	FR	F	$(FR)(RR)$	RR	RRR	$RRRR$	R
$(FR)(RR)$	$(FR)(RR)$	$F(RR)$	FR	F	R	RR	RRR	$RRRR$

Problem 1. Consider the subgroup $H = \{RRRR, F\}$ of the cross symmetries group. Construct the left cosets for this subgroup. How many *different* left cosets are there?

Problem 2. Consider the subgroup $J = \{RRRR, RR, FR, (FR)(RR)\}$ of the cross symmetries group. Construct the left cosets for this subgroup. How many *different* left cosets are there?

Now, let's think about the binary operation $\otimes$ we defined on the set of preimages. Is there some way to describe this operation so that the definition *does not* rely on knowing the group homomorphism; in particular, can we define it solely in terms of left cosets?

Problem 3. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $\boldsymbol{\mathcal{Y}} = (Y,\diamond)$ be another group such that $f : X \to Y$ is a group homomorphism from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}$. If $a \in \text{Pre}_f(u)$ and $b \in \text{Pre}_f(v)$, prove that

$$\text{Pre}_f(u) \otimes \text{Pre}_f(v) = (a * b)\ \ker(f)$$

Problem 3 suggests a way to define a binary operation on the left cosets of a subgroup.

> ***Coset Multiplication***
>
> Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let H be any subgroup of $\boldsymbol{\mathcal{X}}$. For all $a\ H, b\ H \in LC_H$, let $a\ H \circledast b\ H = (a * b)\ H$.

Let's think about whether or not coset multiplication constitutes a group operation on the set of left cosets of a subgroup. To be specific, let's consider the subgroup $H = \{RRRR, F\}$ whose left cosets you computed in Problem 1. We know that H generates four distinct left cosets, namely

$$H = \{RRRR, F\} = RRRR\ H = F\ H \qquad R\ H = \{R, (FR)(RR)\} = (FR)(RR)\ H$$
$$RR\ H = \{RR, FRR\} = FRR\ H \qquad FR\ H = \{FR, RRR\} = RRR\ H$$

Problem 4. Fill in the table below.

$\circledast$	H	$R\ H$	$RR\ H$	$RRR\ H$
H				
$R\ H$				
$RR\ H$				
$RRR\ H$				

Do the left cosets of H form a group under coset multiplication? Explain.

Problem 4. There are *two* ways of representing each member of LC_U. For example, $R\ H = (FR)(RR)\ H$. Do you think the table above would change if we changed some of the representations?

Let's give it a try.

$\circledast$	H	$\boldsymbol{FRRR\ H}$	$RR\ H$	$RRR\ H$
H				
$\boldsymbol{FRRR\ H}$				
$RR\ H$				
$RRR\ H$				

Do the left cosets of H form a group under coset multiplication? Explain.

Problem 4 shows that we can replace the set $R\ H$ with the equivalent set $(FR)(RR)\ H$, and this replacement *changes* the outcome in the tables above in a way that is *not equivalent*. This is a serious problem --- it tells us that coset multiplication is *not a true binary operation*.

This is particularly troubling, since Problem 3 tells us the preimage group operation $\otimes$ can be cast as coset multiplication. Is everything we did in Investigation 11 actually *wrong*?

Problem 5. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $\boldsymbol{\mathcal{Y}} = (Y,\diamond)$ be another group such that $f : X \to Y$ is a group homomorphism from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}$. Prove that $a * u * a^{-1} \in \ker(f)$ for all $a \in X$ and $u \in \ker(f)$.

Problem 6. Consider the subgroup $H = \{RRRR, F\}$ we used in Problems 3 and 4. Show by example that there exist $a \in S_\times$ and $u \in H$ such that $a * u * a^{-1}$ is not a member of H.

> ***Normal Subgroup of a Group***
>
> Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let H be any subgroup of $\boldsymbol{\mathcal{X}}$. We say that H is *normal* provided $a * u * a^{-1} \in H$ for all $a \in X$ and $u \in H$.

It turns out that "normality" is the missing property --- coset multiplication does indeed form a binary operation on the collection of left cosets when the subgroup generating them is normal.

> ***Theorem 12.1***
>
> Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let H be any subgroup of $\boldsymbol{\mathcal{X}}$. If the subgroup H is normal then coset multiplication is a binary operation on LC_H.

We will prove this theorem in the following exercises.

Problem 7. First, suppose that H is a normal subgroup of the group $\boldsymbol{\mathcal{X}}$. Let $m\,H, n\,H \in LC_H$, and suppose that $m\,H = x\,H$ and $n\,H = y\,H$. We need to prove that $(m * n)\,H = (x * y)\,H$.

Part (a). Why do we know that there exist $u, v \in H$ such that $m = x * u$ and $n = y * v$?

Part (b). Why do we know $y^{-1} * u * y \in H$?

Part (c). Use Part (b) to explain why there exist $w \in H$ such that $y * w = u * y$.

Part (d). Use Part (c) to prove that $a * b \in (x * y)\,H$.

Part (e). Use Part (d) to prove that $(a * b)\,H \subseteq (x * y)\,H$.

The proof that $(x * y)\,H \subseteq (a * b)\,H$ is identical to the argument outlined in Parts (a) – (e).

Thanks to Theorem 12.1, the results on preimage groups from Investigation 11 are okay since the kernel of a group homomorphism is a normal subgroup of the domain group.

Problem 8. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and suppose that H is any subgroup such that coset multiplication on the set LC_H is a binary operation. (Normal subgroups would be an example.)

Part (a). Prove that the coset multiplication operation $\circledast$ is associative.

Part (b). Let ε be the identity of the group $\boldsymbol{\mathcal{X}}$, and show that $\varepsilon\,H$ serves as the identity for the algebra $(LC_H,\circledast)$.

Part (c). Complete the proof that the algebra $(LC_H,\circledast)$ is a group.

This group is called the *coset* group for $\boldsymbol{\mathcal{X}}$ *generated* by H. It is also commonly called the *quotient* group generated by H. It is usually denoted by the symbol $\boldsymbol{\mathcal{X}}/H$.

Homework.

Problem 1. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let H be any subgroup of $\boldsymbol{\mathcal{X}}$.

Part (a). If $a \in X$, explain why $a \in a\,H$.

Part (b). If $x \in a\,H$, prove that $x\,H = a\,H$.

Problem 2. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let H be any subgroup of $\boldsymbol{\mathcal{X}}$. Use Problem 1 to help prove the following result:

- If $a\,H \neq b\,H$, then $a\,H \cap b\,H = \emptyset$.

These two problems tell us that every member of X appears in *exactly one* left coset of H. (You probably noticed this when constructing left cosets.)

Problem 3. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let H be any subgroup of $\boldsymbol{\mathcal{X}}$. Show that the mapping $f : H \to a\,H$ defined by $f(u) = a * u$ is a bijection.

Problem 2 tells us that when H is *finite*, every left coset of H has exactly the same number of elements as H. (You probably noticed this when constructing left cosets.)

Problem 4. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a finite group containing exactly n elements, and let H be any subgroup of $\boldsymbol{\mathcal{X}}$. Prove that the number of elements in H must be a divisor of n. (This result is known as *LaGrange's Theorem*.)

Problem 5. Why is every subgroup of a commutative group normal?

Problem 6. Consider the group of cross symmetries.

Part (a). Show that $H = \{RRRR, RR\}$ is a normal subgroup of $\mathcal{S}_\times$.

Part (b). Construct the left cosets for H.

Part (c). Construct the operation table for the quotient group $\mathcal{S}_\times / H = (LC_H, \circledast)$.

Problem 7. Consider the group $\boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_8$ along with the subgroup

$$J = \{(0,0), (2,4), (2,6), (2,2), (0,4), (2,0)\}$$

Part (a). Construct the left cosets for H.

Part (b). Construct the operation table for the quotient group $(\boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_8)/J = \left(LC_J, \circledast\right)$.

Consider the General Linear Group $\boldsymbol{GL_2} = (U_{2\times 2},*)$ of 2×2 invertible matrices with real number entries, under the operation of matrix multiplication.

Problem 8. Consider the set

$$\Theta = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}' \right\}$$

Part (a). Prove that Θ is a subgroup of the General Linear Group.

Part (b). If $A \in \Theta$ and $M \in U_{2\times 2}$, prove $A = M * A * M^{-1}$. (Hence, Θ is a normal subgroup of $\boldsymbol{GL_2}$.)

Part (c). If $M \in U_{2\times 2}$, show that $M\,\Theta = \{aM : a \in \mathbb{R}'\}$.

Part (d). Show by example there exist left cosets $M\,\Theta$ and $N\,\Theta$ such that $M\,\Theta \circledast N\,\Theta \neq N\,\Theta \circledast M\,\Theta$. (Hence, the quotient group $\boldsymbol{GL_2}/\Theta$ is not commutative.)

Problem 9. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let H be any subgroup of $\boldsymbol{\mathcal{X}}$. For all $a, b \in X$, suppose that $(a * b)H = (x * y)H$ for all $x \in aH$ and $y \in bH$.

Part (a). Suppose $u \in H$. Explain why we know $(a^{-1} * a)H = uH$.

Part (b). Use Part (a) to show that there exist $j \in H$ such that $j = a * u * a^{-1}$.

Part (c). Explain why we may conclude that H is normal.

(This exercise shows that the *converse* of Theorem 12.1 is also true; that is, normal subgroups are the *only* subgroups of a group for which coset multiplication is a binary operation.)

Problem 10. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let H be any normal subgroup of $\boldsymbol{\mathcal{X}}$.

Part (a). Show that the function $\eta_H : X \to LC_H$ defined by $\eta_H(a) = a\,H$ is a group homomorphism from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{X}}/H$.

Part (b). Show that $\ker(\eta_H) = H$. (This tells us that *every* normal subgroup of a group is the kernel of a group homomorphism.)

Problem 10 also tells us that quotient groups and preimage groups are actually the same. In particular, if $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and H is any normal subgroup of $\boldsymbol{\mathcal{X}}$, then $\eta_H : X \to LC_H$ is a group epimorphism such that $\ker(\eta_H) = H$, and $P_{\eta_H} = \left\{\mathrm{Pre}_{\eta_H}(a\,H) : a\,H \in LC_H\right\} = \{a \ker(\eta_H) : a \in X\}$.