Let X be any nonempty set. We will let $[X \to X]$ represent the set of all functions $f: X \to X$. Now, if $f, g \in [X \to X]$, we can talk about the *composition* of these functions. Remember, the functions $f \circ g$ and $g \circ f$ are defined by the formulas

$$[f \circ g\,](x) = f\big(g(x)\big) \qquad\qquad [g \circ f\,](x) = g\big(f(x)\big)$$

Function composition serves as a binary operation on the set $[X \to X]$.

When we want to define functions whose domain is a finite set, we commonly use *tabular notation* as a shorthand way to do so. For example, suppose we let $X = \{1,2,3,4\}$. Consider the function $h: X \to X$ defined by

$$h(1) = 1 \quad h(2) = 1 \quad h(3) = 4 \quad h(4) = 4$$

It is common to write this function assignment as

$$h\colon \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}$$

In this notation, the top row represents the domain of the function, and the bottom row represents the range.

We can compute the composition of two functions from $[X \to X]$ using tabular notation as well. We only need to remember that function composition is read *from right to left*. Suppose $f, g \in [X \to X]$ are defined by

$$f\,:\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \quad g\,:\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

The composition $g \circ f$ would be

Move right to left

$$g \circ f\,:\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

In words, we know that $f(1) = 3$, and we know that $g(3) = 2$. Therefore, we know $g\big(f(1)\big) = 2$. Each column of the tabular notation for $g \circ f$ is filled in this way.

Problem 1. Let $X = \{1,2,3,4\}$ and consider the functions f, g, and h written in tabular notation above.

Part (a). Use tabular notation to construct the composite functions $f \circ g$ and $h \circ f$.

Part (b). Is function composition a commutative operation on the set $[X \to X]$? Explain.

THEOREM 2.1
Let X be any nonempty set, and consider the family of functions $[X \to X]$. If $f, g, h \in [X \to X]$, then it is always true that $f \circ (g \circ h) = (f \circ g) \circ h$. In other words, the binary operation of function composition on the set $[X \to X]$ is always associative.

Proof. We need to show that, for all $x \in X$, we have $[f \circ (g \circ h)](x) = [(f \circ g) \circ h](x)$. To this end, observe that

$$[f \circ (g \circ h)](x) = f\big([g \circ h](x)\big) = f\big(g\big(h(x)\big)\big)$$

$$[(f \circ g) \circ h](x) = [(f \circ g)]\big(h(x)\big) = f\big(g\big(h(x)\big)\big)$$

QED

Let X and Y be any nonempty sets, and suppose that $f : X \to Y$ is a function. We say that f is an *injection* provided $f(a) = f(b)$ always implies that $a = b$. (In other words, the equation $v = f(u)$ has at at most one solution for every $v \in Y$.)

Problem 2. Let X, Y, and Z be any nonempty sets. Suppose that $f : Y \to Z$ and $g : X \to Y$ are both injections.

Part (a). Suppose that $u, v \in X$ are such that $f\big(g(u)\big) = f\big(g(v)\big)$. Must it be true that $u = v$? Explain your thinking.

Part (b). Is it true that the composite function $f \circ g$ is an injection? Explain.

Let X and Y be any nonempty sets, and suppose that $f : X \to Y$ is a function. We way that f is a *surjection* provided, for every $b \in Y$ there is at least one $a \in X$ such that $b = f(a)$. (In other words, the equation $b = f(a)$ has at least one solution for every $b \in Y$.)

Problem 3. Let $J = \{1,2,3,4,5\}$, let $K = \{a, b, c\}$ and let $L = \{u, v, w\}$ and consider the following surjections $f : K \to L$ and $g : J \to K$.

$$f : \begin{pmatrix} a & b & c \\ v & w & u \end{pmatrix} \quad g : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & b & c & a & a \end{pmatrix}$$

Part (a). How many solutions do each of the following equations have in the set J? Explain your strategy.

$$u = [f \circ g](x) \qquad v = [f \circ g](x) \qquad w = [f \circ g](x)$$

Part (b). Is the composite function $f \circ g$ a surjection? Explain.

Problem 4. Let X, Y, and Z be any nonempty sets. Suppose that $f : Y \to Z$ and $g : X \to Y$ are both surjections.

Part (a). Suppose that $a \in Z$. Explain why the equation $a = f(y)$ has a solution in the set Y.

Part (b). Let $b \in Y$ be any solution to the equation $a = f(y)$. Explain why the equation $b = g(x)$ has a solution in the set X.

Part (c). Explain why Parts (a) and (b) tell us that the composite function $f \circ g$ is a surjection.

Problem 5. Let X be any nonempty set, and let Bi$[X \to X]$ be the set of all *bijections* (functions which are injections and surjections) from X to X. Explain why the set Bi$[X \to X]$ is closed under function composition.

In light of Problem 5, we know $\mathbf{Bi}[X \to X] = (\mathrm{Bi}[X \to X], \circ)$ is an algebra. When X is a finite nonempty set, the members of Bi$[X \to X]$ are usually called *permutations* on X, because they simply rearrange the members of the set.

If $X = \{1,2,3 \dots, n\}$ then it is common to let $\mathcal{P}_n$ denote the set Bi$[X \to X]$. We will let $\boldsymbol{\mathcal{P}_n} = (\mathcal{P}_n, \circ)$ represent the algebra of permutations on X under function composition.

Problem 6. Consider the permutation algebra $\boldsymbol{\mathcal{P}_4}$ and the bijections

$$\alpha : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad \beta : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Compute the function composition $\alpha \circ \beta$ and $\beta \circ \alpha$. Write your answer in tabular notation.

Homework.

Problem 1. Let $X = \mathbb{R} - \{0,1\}$ and consider the following members of $\text{Bi}[X \to X]$.

$$\varepsilon(x) = x \qquad q(x) = 1 - x \qquad r(x) = \frac{1}{x} \qquad s(x) = \frac{1}{1-x} \qquad t(x) = \frac{x}{x-1} \qquad u(x) = \frac{x-1}{x}$$

Part (a). Fill in the table below. (Here $\circ$ represents function composition.)

$\circ$	$\boldsymbol{\varepsilon}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{s}$	$\boldsymbol{t}$	$\boldsymbol{u}$
$\boldsymbol{\varepsilon}$						
$\boldsymbol{q}$						
$\boldsymbol{r}$						
$\boldsymbol{s}$						
$\boldsymbol{t}$						
$\boldsymbol{u}$						

Part (b). Let $CR = \{\varepsilon, q, r, s, t, u\}$. (This is traditionally called the set of *cross-ratios*.) Does the pair $(CR, \circ)$ represent an algebra? Explain.

Problem 2. Consider the set $X = \{1,2,3\}$. There are six elements in the permutation algebra $\boldsymbol{\mathcal{P}_3}$, namely

$$\varepsilon_3 : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \alpha : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad \beta : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\gamma : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad \delta : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \zeta : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Fill in the operation table below. (The symbol $\circ$ denotes function composition.)

$\circ$	$\boldsymbol{\varepsilon_3}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\zeta}$
$\boldsymbol{\varepsilon_3}$						
$\boldsymbol{\alpha}$						
$\boldsymbol{\beta}$						
$\boldsymbol{\gamma}$						
$\boldsymbol{\delta}$						
$\boldsymbol{\zeta}$						

Problem 3. Consider the permutation algebra $\boldsymbol{\mathcal{P}_6}$ and the bijections

$$\alpha : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 3 & 4 & 6 \end{pmatrix} \qquad \varepsilon_6 : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Consider the equation $\alpha \circ x = \varepsilon_6$. Is there a solution to this equation in the algebra $\boldsymbol{\mathcal{P}_6}$?

Let X and Y be nonempty sets, and let $f : X \to Y$ and $g : Y \to X$ be functions. We say that f and g are *function inverses* provided $[f \circ g](u) = u$ for all $u \in Y$ and $[g \circ f](v) = v$ for all $v \in X$. Function inverses "reverse" each other.

Problem 4. Consider the set of six functions from Problem 1.

Part (a). Use the operation table you created to identify the function inverse for t. Explain your strategy.

Part (b). Does every member of the set CR have a function inverse? How do you know?

Problem 5. Consider the six permutations from Problem 2. Which of these permutations has a function inverse? What is the function inverse for each of those that do?

Problem 6. Consider the function f from the permutation algebra $\boldsymbol{\mathcal{P}_8}$ defined by

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 8 & 1 & 3 & 2 & 7 & 4 \end{pmatrix}$$

Use tabular notation to write down the function inverse for f. Explain the strategy you used.