In this investigation, we will introduce the central concept that we will be studying in this course.

Problem 1. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is an algebra. An element $u \in X$ is called an *identity* for the algebra $\boldsymbol{\mathcal{X}}$ provided $a * u = a$ and $u * a = a$ for all $a \in X$.

If ε and δ are both identity elements for the algebra $\boldsymbol{\mathcal{X}}$, prove that $\varepsilon = \delta$. (Hint: Think about $\varepsilon * \delta$.)

Problem 2. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is an algebra with identity ε. An element $a \in X$ has an *inverse* in the algebra $\boldsymbol{\mathcal{X}}$ provided there exists at least one $b \in X$ such that $a * b = \varepsilon$ and $b * a = \varepsilon$.

Suppose b and c are both inverses for an element $a \in X$. If the operation * is *associative*, prove $b = c$.
Hint: First explain why we know $b = (c * a) * b$.

Problem 3. Let $X = \{a, b, c, d\}$ and consider the algebra $\boldsymbol{\mathcal{X}} = (X,*)$ defined by the operation table below.

$*$	***a***	***b***	***c***	***d***	***e***
a	*d*	*b*	*c*	*a*	*e*
b	*d*	*d*	*a*	*b*	*c*
c	*c*	*e*	*d*	*c*	*d*
d	*a*	*b*	*c*	*d*	*e*
e	*b*	*d*	*d*	*e*	*a*

Part (a). This algebra has an identity. What is it?

Part (b). Does every element possess an inverse in this algebra? Do any elements have more than one inverse?

Group

Suppose that X is a any nonempty set endowed with a binary operation *. We say that the algebra $\boldsymbol{\mathcal{X}} = (X,*)$ is a *group* provided the following conditions are met.

1. The operation * is associative.
2. There is an identity for $\boldsymbol{\mathcal{X}}$.
3. Every member of X has an inverse in the algebra $\boldsymbol{\mathcal{X}}$.

Problem 4. Is the algebra in Problem 3 a group? Explain your thinking.

Problem 5. Let $X = \mathbb{R} - \{0,1\}$ and consider the subset $CR = \{\varepsilon, q, r, s, t, u\}$ of Bi$[X \to X]$ where

$$\varepsilon(x) = x \qquad q(x) = 1 - x \qquad r(x) = \frac{1}{x} \qquad s(x) = \frac{1}{1-x} \qquad t(x) = \frac{x}{x-1} \qquad u(x) = \frac{x-1}{x}$$

In the homework for Investigation 2, you constructed the operation table for the cross-ratio algebra $(CR,\circ)$ where $\circ$ represents function composition. Here is the operation table you constructed.

$\circ$	ε	q	r	s	t	u
ε	ε	q	r	s	t	u
q	q	ε	u	t	s	r
r	r	s	ε	q	u	t
s	s	r	t	u	q	ε
t	t	u	s	r	ε	q
u	u	t	q	ε	r	s

Is the algebra $(CR,\circ)$ a group? Explain your answer.

Problem 6. In the homework for Investigation 4, you constructed the operation table for the algebra $\boldsymbol{\mathcal{Z}}_6^m = (\mathbb{Z}_6, \boxtimes_6)$. Here is the operation table you constructed.

$\boxtimes_6$	**0**	**1**	**2**	**3**	**4**	**5**
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Is the algebra $\boldsymbol{\mathcal{Z}}_6^m$ a group? Explain your thinking.

Problem 7. Here is the operation table for the symmetry algebra $\boldsymbol{S}_\Delta = (S_\Delta, *)$ from Investigation 3.

$*$	RRR	RR	R	F	FR	$F(RR)$
RRR	RRR	RR	R	F	FR	$F(RR)$
RR	RR	R	RRR	FR	$F(RR)$	F
R	R	RRR	RR	$F(RR)$	F	FR
F	F	$F(RR)$	FR	RRR	R	RR
FR	FR	F	$F(RR)$	RR	RRR	R
$F(RR)$	$F(RR)$	FR	F	R	RR	RRR

Is the algebra $\boldsymbol{S}_\Delta$ a group? Explain your thinking.

Problem 8. Let $n > 1$ be a fixed positive integer, and let $n\mathbb{Z} = \{nx : x \in \mathbb{Z}\}$.

Part (a). Show that this set is closed under integer addition.

Part (b). Does the algebra $n\boldsymbol{\mathcal{Z}} = (n\mathbb{Z}, +)$ have an identity?

Part (c). Is the algebra $n\boldsymbol{\mathcal{Z}} = (n\mathbb{Z}, +)$ a group? Justify your answer.

Problem 9. Let $\mathbb{R}^*$ denote the set of nonzero real numbers, and consider the binary rule $\sqcap$ defined by

$$a \sqcap b = \frac{ab}{4}$$

Part (a). If we want to decide whether or not the algebra $\boldsymbol{\mathcal{R}}^* = (\mathbb{R}^*, \sqcap)$ is a group, what would we need to check?

Part (b). Prove that the algebra $\boldsymbol{\mathcal{R}}^* = (\mathbb{R}^*, \sqcap)$ is a group.

Special Notation for Inverses

Suppose that $\boldsymbol{\mathcal{X}} = (X, *)$ is a group. If $a \in X$, then we know that a has exactly one inverse in the algebra $\boldsymbol{\mathcal{X}}$. It is traditional to let a^{-1} represent the inverse of a with respect to the operation $*$.

Homework.

Problem 1. Let $S = \{5, 15, 25, 35\}$.

Part (a). Fill in the table below.

$\boxtimes_{40}$	5	15	25	35
5				
15				
25				
35				

Part (b). Does the set S form a group under multiplication modulo 40? Explain how you decided.

Problem 2. Two integers are *relatively prime* provided their greatest common divisor is 1. Let $\mathbb{U}_{12}$ represent the subset of $\mathbb{Z}_{12}$ containing those integers relatively prime to 12. Is $\mathbb{U}_{12}$ a group under multiplication modulo 12? Justify your answer.

Problem 3. For a fixed positive integer $n > 1$, consider the algebra $\boldsymbol{\mathcal{Z}}_n = (\mathbb{Z}_n, \boxplus_n)$.

Part (a). Show that 0 serves as the identity for this algebra.

Part (b). If $a \in \mathbb{Z}_n$ is nonzero, show that $n - a$ is the inverse for a in this algebra.

Problem 4. Consider the algebra $\mathbf{Bi}[X] = (\mathrm{Bi}[X \to X], \circ)$ introduced in Investigation 2.

Part (a). Show that the function $\varepsilon: X \to X$ defined by $\varepsilon(x) = x$ serves as the identity element for the algebra $\mathbf{Bi}[X]$.

Part (b). Let $f \in \mathrm{Bi}[X \to X]$ and consider the function $g \in \mathrm{Bi}[X \to X]$ defined by the formula

$$g(y) = x \quad \Leftrightarrow \quad y = f(x)$$

Show that g serves as the inverse for the function f in the algebra $\mathbf{Bi}[X]$.

Group of Bijections on a Set X

If X is any nonempty set, then the collection $\mathrm{Bi}[X \to X]$ of all bijections on X forms a group under function composition.

Problem 5. Let $X = \{1,2,3,4\}$ and consider the permutation algebra $\boldsymbol{\mathcal{P}_4} = (\mathcal{P}_4, \circ)$. We know this algebra is a group. Let

$$f\colon \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad g\colon \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Part (a). Write the inverse for f and the inverse for g in the group $\boldsymbol{\mathcal{P}_4}$ using tabular notation. How did you determine the formula for the inverses?

Part (b). Use tabular notation to construct the formula for the following members of $\mathcal{P}_4$.

$$f^{-1} \circ g \circ f \qquad \text{and} \qquad g \circ f \circ g^{-1}$$

Problem 6. Prove that the set $X = \mathbb{R} - \{-1\}$ forms a group under the binary operation

$$a \boxminus b = a + ab + b$$

Is this group commutative?

Problem 7. Let $\mathbb{R}'$ represent the set of all positive real numbers except for 1. Consider the binary operation $\diamond$ defined on $\mathbb{R}'$ according to the rule

$$a \diamond b = a^{\ln(b)}$$

Part (a). Show that the algebra $\mathcal{R}^{\diamond} = (\mathbb{R}', \diamond)$ has an identity.

Part (b). Show that every member of the algebra $\mathcal{R}^{\diamond} = (\mathbb{R}', \diamond)$ has in inverse.

Part (c). Is the algebra $\mathcal{R}^{\diamond} = (\mathbb{R}', \diamond)$ a group? Justify your answer.