In this investigation, you will practice working with groups and some of their properties.

**Problem 1.** Consider the permutation Consider the permutation group $\boldsymbol{\mathcal{P}_4}$ and the bijections

$$\alpha : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad \beta : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

**Part (a).** Compute the function composition $\alpha \circ \beta$.

**Part (b).** Construct the group inverse for each of the functions $\alpha, \beta$ and $\alpha \circ \beta$.

**Part (c).** Is it possible to write $(\alpha \circ \beta)^{-1}$ as a composition of $\alpha^{-1}$ and $\beta^{-1}$?

**Problem 2.** Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $a, b \in X$.

**Part (a).** What can you say about $(b^{-1} * a^{-1}) * (a * b)$? Justify your answer using the properties of groups.

**Part (b).** How is $(a * b)^{-1}$ related to $a^{-1}$ and $b^{-1}$? Justify your answer using the properties of groups.

**Problem 3.** Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $a, b \in X$. Prove that $(a * b)^{-1} = a^{-1} * b^{-1}$ if and only if $a * b = b * a$.

***Powers of Group Elements***

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, let $a \in X$, and let $n$ be a positive integer. We define the element $a^n$ to be the result of applying $a$ to itself $n-1$ times under the operation $*$. In symbols, we let

$$a^n = \underbrace{a * a * \ldots * a}_{n \text{ times}}$$

In the spirit of this definition, we let $a^{-n} = (a^n)^{-1}$ and let $a^0$ be the identity element.

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $a \in X$. We can use the power definition above, along with mathematical induction, to show that $a^m * a^n = a^{m+n}$ for *any* integers $m$ and $n$. This process is tedious, however; and will be omitted.

**Problem 4.** What is the value of $3^4$ in each of the following groups?

**Part (a).** The group $\boldsymbol{\mathcal{Z}}$ of integers under addition

**Part (b).** The group $\boldsymbol{\mathcal{Z}}_4 = (\mathbb{Z}_4, \boxplus_4)$

**Part (c).** The group $\boldsymbol{\mathcal{U}}_{20} = (\mathbb{U}_{20}, \boxtimes_{20})$ (See Investigation 5 Problem 2.)

**Problem 5.** Consider the permutation group $\boldsymbol{\mathcal{P}_4}$ and the permutation

$$\alpha : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

**Part (a).** Compute the powers $\alpha^1$ through $\alpha^6$. What patterns do you notice?

**Part (b).** Compute the powers $\alpha^{-1}$ through $\alpha^{-6}$. What patterns do you notice?

**Problem 6.** Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $a \in X$. Consider the set $\mathrm{Pow}[a] = \{a^n : n \in \mathbb{Z}\}$.

**Part (a).** Prove that $\mathrm{Pow}[a]$ is closed under the group operation $*$.

**Part (b).** Show that the algebra $(\mathrm{Pow}[a],*)$ has an identity.

**Part (c).** Show that every element of $\mathrm{Pow}[a]$ has an inverse which is also a member of $\mathrm{Pow}[a]$.

**Part (d).** Is the algebra $(\mathrm{Pow}[a],*)$ a group? Explain.

**Homework.**

**Problem 1.** Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group and let $a \in X$. Explain why we know that $a = (a^{-1})^{-1}$.

**Problem 2.** Consider the group $\boldsymbol{\mathcal{S}}_\times = (S_\times,*)$ of cross symmetries. Is it true that $(F * R)^2 = F^2 * R^2$ in this group?

**Problem 3.** Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group and let $a, b, c \in X$. Use the properties of groups to prove the following.

- If $a * b = a * c$, then $b = c$.
- If $b * a = c * a$, then $b = c$.

This is referred to as the *cancellation property* for groups.

**Problem 4.** Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group and let $a, b \in X$. Use the fact that $(a * b)^2 = (a * b) * (a * b)$ to help you prove that $(a * b)^2 = a^2 * b^2$ implies that $a * b = b * a$.

Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group with identity $\varepsilon$, and suppose that $a \in X$. We say that $a$ has *finite order* in this group provided there exists a smallest positive integer $p$ such that $a^p = \varepsilon$.

**Problem 5.** Determine the finite order of each element below in the specified group.

**Part (a).** The element 4 in the group $\boldsymbol{\mathcal{Z}}_6 = (\mathbb{Z}_6, \boxplus_6)$

**Part (b).** The element 7 in the group $\boldsymbol{\mathcal{U}}_{20} = (\mathbb{U}_{20}, \boxtimes_{20})$

**Part (c).** The permutation $\sigma$ given below in the permutation group $\boldsymbol{\mathcal{P}}_5$

$$\sigma : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \end{pmatrix}$$

**Problem 6.** Does the element 2 have finite order in the group $\boldsymbol{\mathcal{Z}}$ of integers under addition? Explain.

**Problem 7.** In the group $\boldsymbol{\mathcal{U}}_{20} = (\mathbb{U}_{20}, \boxtimes_{20})$, what are the elements of the set Pow[7]?

**Problem 8.** In the permutation group $\boldsymbol{\mathcal{P}_5}$, what are the elements of the set Pow[$\sigma$], where $\sigma$ is defined in Problem 5 (c)?

The set $M_2$ of all $2 \times 2$ invertible real-valued matrices forms a group under matrix multiplication. (You actually proved this in linear algebra.) This group is called the *general linear group*; we will let $\boldsymbol{GL_2} = (M_2, *)$ represent this group.

**Problem 9.** In the general linear group, describe the elements of the set Pow[$A$] if we let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

**Problem 10.** In the general linear group, describe the elements of the set Pow[$B$] if we let

$$B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

**Problem 11.** Suppose that $\boldsymbol{\mathcal{X}} = (X, *)$ is a group, and suppose that $a \in X$. Use the method of mathematical induction to prove the following result.

- If $n$ is any positive integer, then $(a^{-1})^n$ serves as the group inverse for $a^n$.