In the last few investigations, you may have noticed that many of the algebras we have constructed share a lot of the same structural features. In this investigation, we will explore this fact.

Problem 1. Consider the general linear group $\boldsymbol{GL_2} = (M_2,*)$ of 2×2 invertible matrices under matrix multiplication. In particular, consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

Part (a). The set $\mathrm{Pow}[A]$ contains six distinct members. Determine these members.

$A^2 =$ $\quad$ $A^3 =$ $\quad$ $A^4 =$

$A^5 =$ $\quad$ $A^6 =$

Part (b). Use your matrices to fill in the operation table for the algebra $(\mathrm{Pow}[A]\,,*)$.

$*$	A	A^2	A^3	A^4	A^5	A^6
A						
A^2						
A^3						
A^4						
A^5						
A^6						

Part (c). Did you really need the formulas for the powers of A from Part (a) in order to fill in this table? Explain your thinking.

Part (d). Do you think the algebra $(\mathrm{Pow}[A]\,,*)$ has the same "structure" as the group $\boldsymbol{\mathcal{Z}}_6 = (\mathbb{Z}_6, \boxplus_6)$? What leads you to your conclusion?

Problem 2. The operation tables for the triangle symmetries group $\boldsymbol{S}_\Delta = (S_\Delta,*)$ and cross-ratio group $(CR,\circ)$ are shown below.

Part (a). In the empty table provided, rearrange the elements of the cross-ratio group so that the patterns presented in its table exactly match the patterns in the symmetries table.

$*$	$\boldsymbol{RRR}$	$\boldsymbol{RR}$	$\boldsymbol{R}$	$\boldsymbol{F}$	$\boldsymbol{FR}$	$\boldsymbol{FRR}$
$\boldsymbol{RRR}$	RRR	RR	R	F	FR	FRR
$\boldsymbol{RR}$	RR	R	RRR	FR	FRR	F
$\boldsymbol{R}$	R	RRR	RR	FRR	F	FR
$\boldsymbol{F}$	F	FRR	FR	RRR	R	RR
$\boldsymbol{FR}$	FR	F	FRR	RR	RRR	R
$\boldsymbol{FRR}$	FRR	FR	F	R	RR	RRR

$\circ$	$\boldsymbol{\varepsilon}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{s}$	$\boldsymbol{t}$	$\boldsymbol{u}$
$\boldsymbol{\varepsilon}$	ε	q	r	s	t	u
$\boldsymbol{q}$	q	ε	u	t	s	r
$\boldsymbol{r}$	r	s	ε	q	u	t
$\boldsymbol{s}$	s	r	t	u	q	ε
$\boldsymbol{t}$	t	u	s	r	ε	q
$\boldsymbol{u}$	u	t	q	ε	r	s

$\circ$						

Part (b). Explain the strategies you used to determine your arrangement.

Isomorphic Groups

Suppose that $\boldsymbol{X} = (X,*)$ and $\boldsymbol{Y} = (Y,\odot)$ are groups. We say these groups are *isomorphic* provided they have exactly the same structure. The group $\boldsymbol{Y}$ is isomorphic to $\boldsymbol{X}$ when it is possible to arrange the members of Y so that they relate to each other under the $\odot$ operation in exactly the same way as the members of X under the $*$ operation.

The triangle symmetries and cross-ratio groups are isomorphic, as are the groups $\boldsymbol{\mathcal{Z}}_6$ and $(\text{Pow}[A]\ ,*)$ from Problem 1.

Problem 3. Do you think that the triangle symmetries group $\boldsymbol{S}_\Delta = (S_\Delta, *)$ is isomorphic to the group $\boldsymbol{\mathcal{Z}}_6 = (\mathbb{Z}_6, \boxplus_6)$? Explain your thinking.

Problem 4. In Homework Problem 5 of Investigation 3 you worked with the rectangle symmetries group. The operation table for this group along with the operation table for $\boldsymbol{\mathcal{Z}}_4$ is shown below.

$*$	**R**	**F**	**RR**	**RF**
R	RR	RF	R	F
F	RF	RR	F	R
RR	R	F	RR	RF
RF	F	R	RF	RR

$\boxplus_4$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Are these groups isomorphic? Explain your thinking.

Problem 5. Think about the rearrangement of the cross-ratio set CR you constructed in Problem 2. In a sense, you "renamed" the members of the set CR as certain members of the set S_Δ.

Part (a). Define a function $f : CR \to S_\Delta$ which accomplishes your "renaming."

$f(\varepsilon) =$ $\quad$ $f(q) =$ $\quad$ $f(r) =$ $\quad$ $f(s) =$ $\quad$ $f(t) =$ $\quad$ $f(u) =$

Part (b). Are any of the following equations valid?

$f(r \circ t) = f(r) * f(t)$ $\quad$ $f(u \circ q) = f(u) * f(q)$ $\quad$ $f(s \circ r) = f(s) * f(r)$

$f(s^3) = [f(s)]^3$ $\quad$ $f(u^{-1}) = [f(u)]^{-1}$

Isomorphism Between Groups

Suppose that $\boldsymbol{\mathcal{X}} = (X, *)$ and $\boldsymbol{\mathcal{Y}} = (Y, \odot)$ are groups. An *isomorphism between* $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ is a bijection $f : X \to Y$ with the property that $f(a * b) = f(a) \odot f(b)$ for all $a, b \in X$. Two groups are isomorphic if and only if there is an isomorphism between them.

It is common to say that an isomorphism between groups "preserves" the group operations.

Problem 6. Consider the bijections $g : \mathrm{Pow}[A] \to \mathbb{Z}_6$ and $h :\ \mathrm{Pow}[A] \to \mathbb{Z}_6$ defined below.

$$g(A) = 5 \qquad g(A^2) = 2 \qquad g(A^3) = 3 \qquad g(A^4) = 4 \qquad g(A^5) = 1 \qquad g(A^6) = 0$$

$$h(A) = 5 \qquad h(A^2) = 4 \qquad h(A^3) = 3 \qquad h(A^4) = 2 \qquad h(A^5) = 1 \qquad h(A^6) = 0$$

Is either of these functions an isomorphism? Justify your answer.

Problem 7. Is the group $\boldsymbol{\mathcal{Z}} = (\mathbb{Z}, +)$ of integers isomorphic to the group $\mathbf{2}\boldsymbol{\mathcal{Z}} = (2\mathbb{Z}, +)$ of even integers? Justify your answer.

Problem 8. Let $\boldsymbol{\mathcal{R}^p} = (\mathbb{R}^+, \cdot)$ represent the group of positive real numbers under multiplication, and let $\boldsymbol{\mathcal{R}_+} = (\mathbb{R}, +)$ denote the group of real numbers under addition. Consider the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \ln(x)$. Is this function an isomorphism between $\boldsymbol{\mathcal{R}_+}$ and $\boldsymbol{\mathcal{R}^p}$?

Homework.

Problem 1. Let $\mathbb{Q}^*$ represent the set of nonzero rational numbers, and let $\mathbb{Q}^+$ represent the set of positive rational numbers. Consider the groups $\boldsymbol{Q}^* = (\mathbb{Q}^*,\cdot)$ and $\boldsymbol{Q}^+ = (\mathbb{Q}^+,\cdot)$, where $\cdot$ is rational number multiplication, along with the function $f : \mathbb{Q}^* \to \mathbb{Q}^+$ defined by $f(x) = |x|$.

Part (a). Does the function f preserve the group operation?

Part (b). Is the function f an isomorphism between these groups?

Problem 2. Let $\mathbb{R}^*$ denote the set of nonzero real numbers, and consider the binary rule $\sqcap$ defined by

$$a \sqcap b = \frac{ab}{4}$$

In Problem 1 of Investigation 5, you showed that $\boldsymbol{\mathcal{R}}^* = (\mathbb{R}^*,\sqcap)$ is a group. Let $\boldsymbol{\mathcal{R}}. = (\mathbb{R}^*,\cdot)$ represent the group of nonzero real numbers under multiplication. Prove that the function $f : \mathbb{R}^* \to \mathbb{R}^*$ defined by

$$f(x) = 4 \cdot x$$

is an isomorphism between these groups. (You have to be careful about which group f is taking you *from*.)

Problem 3. Consider the general linear group $\boldsymbol{GL_2} = (M_2,*)$ of 2×2 invertible matrices under matrix multiplication. In Homework Problem 9 of Investigation 6, you showed that

$$\text{Pow}\left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right] = \left\{\begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\right\}$$

Prove that the group $\left(\text{Pow}\left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right],*\right)$ is isomorphic to the group $\boldsymbol{\mathcal{Z}}$.

Problem 4. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ are groups, and suppose that $h : X \to Y$ is an isomorphism. Use the function to prove that, if $\boldsymbol{\mathcal{X}}$ is commutative, then $\boldsymbol{\mathcal{Y}}$ must also be commutative.

Problem 5. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ are groups, and suppose that $h : X \to Y$ is an isomorphism. Also, suppose that δ is the identity for the group $\boldsymbol{\mathcal{Y}}$, and suppose that ε is the identity for the group $\boldsymbol{\mathcal{X}}$.

Part (a). Use group properties and the fact that $\varepsilon * \varepsilon = \varepsilon$ to help you prove $h(\varepsilon) = \delta$.

Part (b). Let $a \in X$. Use the fact that $h(\varepsilon) = h(a * a^{-1})$ to help you prove that $h(a^{-1})$ serves as the inverse for $h(a)$ in the group $\boldsymbol{\mathcal{Y}}$.

In light of Problem 5, we say that isomorphisms between groups *preserve* the identity and inverses.

Problem 6. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ are groups, and suppose that $h : X \to Y$ is an isomorphism. Prove that the inverse function $g : Y \to X$ for the function h is also an isomorphism. (Remember, if $u \in Y$, then there exist $a \in X$ such that $u = h(a)$ and $a = g(u)$.)

In the next exercise, we will prove the following result:

- Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group. If $a \in X$ has finite order n, then the group $(\mathrm{Pow}[A],*)$ is isomorphic to the group $\boldsymbol{\mathcal{Z}}_n$.

Problem 7. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group and suppose $a \in X$ has finite order n. Let ε be the identity for the group $\boldsymbol{\mathcal{X}}$.

Part (a). If m is any integer, use the Division Algorithm to help prove that there exists a unique integer $r \in \mathbb{Z}_n$ such that $a^m = a^r$.

Part (b). Suppose $r, s \in \mathbb{Z}_n$ and suppose that $r < s$. Prove that $a^r \neq a^s$. (Assume the contrary and use the definition of finite order.)

Part (a) tells us $\mathrm{Pow}[A] = \{\varepsilon, a, \dots, a^{n-1}\}$, and Part (b) tells us these elements are all distinct. Consequently, $\mathrm{Pow}[A]$ contains the same number of elements as does $\mathbb{Z}_n$. This tells us there is a bijection from $\mathrm{Pow}[A]$ to $\mathbb{Z}_n$. One such bijection is $f : \mathrm{Pow}[A] \to \mathbb{Z}_n$ defined by $f(a^r) = r$.

Part (c). Prove this function is an isomorphism.