In this investigation, we will explore one way to create new groups (sometimes with new properties) from groups we already have. If X and Y are nonempty sets, remember that the *Cartesian products* formed from X and Y are the sets

$$X \times Y = \{(a,b) : a \in X, b \in Y\} \qquad\qquad Y \times X = \{(b,a) : a \in X, b \in Y\}$$

Problem 1. The table below defines a binary operation on the cross product $\mathbb{Z}_4 \times \mathbb{Z}_2$.

$\otimes$	$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{2},\mathbf{0})$	$(\mathbf{2},\mathbf{1})$	$(\mathbf{3},\mathbf{0})$	$(\mathbf{3},\mathbf{1})$
$(\mathbf{0},\mathbf{0})$	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)
$(\mathbf{0},\mathbf{1})$	(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)	(3,1)	(3,0)
$(\mathbf{1},\mathbf{0})$	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)	(0,0)	(0,1)
$(\mathbf{1},\mathbf{1})$	(1,1)	(1,0)	(2,1)	(2,0)	(3,1)	(3,0)	(0,1)	(0,0)
$(\mathbf{2},\mathbf{0})$	(2,0)	(2,1)	(3,0)	(3,1)	(0,0)	(0,1)	(1,0)	(1,1)
$(\mathbf{2},\mathbf{1})$	(2,1)	(2,0)	(3,1)	(3,0)	(0,1)	(0,0)	(1,1)	(1,0)
$(\mathbf{3},\mathbf{0})$	(3,0)	(3,1)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
$(\mathbf{3},\mathbf{1})$	(3,1)	(3,0)	(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)

Part (a). How is the operation $\otimes$ related to addition modulo 4 and addition modulo 2?

Part (b). Is $(\mathbb{Z}_4 \times \mathbb{Z}_2, \otimes)$ a group? Explain your thinking.

Problem 2. What is the finite order of the element $(3,1)$ in this algebra?

Problem 3. The algebra $(\mathbb{Z}_4 \times \mathbb{Z}_2, \otimes)$ is a group. Do you think this group is isomorphic to the group $\boldsymbol{S}_+ = (S_+,*)$ of cross symmetries? Explain your thinking.

Problem 4. Do you think $(\mathbb{Z}_4 \times \mathbb{Z}_2, \otimes)$ is isomorphic to the group $\boldsymbol{\mathcal{Z}}_8 = (\mathbb{Z}_8, \boxplus_8)$? Explain your thinking.

Theorem 7.1 (Products of Groups)

Let $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\diamond)$ be any groups. If we define a binary rule $\otimes$ on the Cartesian product $X \times Y$ by

$$(a,b)\otimes(c,d) = (a * c, b \diamond d)$$

then $X \times Y$ forms a group under $\otimes$. We denote this group by $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}} = (X \times Y, \otimes)$ and call it the *product group of* $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$.

Proof of Theorem 7.1

Problem 5. First, note that since * and $\diamond$ are binary operations on the sets X and Y, respectively, we know that for all $(a,b),(c,d) \in X \times Y$, we must have $a * c \in X$ and $b \diamond d \in Y$. Hence, $(a,b)\otimes(c,d) \in X \times Y$, and we may conclude that $\otimes$ is a binary operation on $X \times Y$. Therefore, we know $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}$ is an algebra.

Part (a). Prove that the operation $\otimes$ is associative.

Part (b). Let ε_X and ε_Y denote the identity elements for $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$, respectively. Complete the proof that $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}$ is a group.

Problem 6. Fill in the operation table for the product group $\boldsymbol{\mathcal{Z}}_2 \times \boldsymbol{\mathcal{Z}}_2$. (This group is called the *Klein Four-Group*.)

$\otimes$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$				
$(0,1)$				
$(1,0)$				
$(1,1)$				

Problem 7. Is the Klein Four-Group isomorphic to the group $\mathcal{Z}_4 = (\mathbb{Z}_4, \boxplus_4)$? Explain your thinking.

Problem 8. Is the Klein Four-Group isomorphic to the group of rectangle symmetries? (See Homework Problem 5 of Investigation 3 and Problem 4 of Investigation 6.)

Problem 9. Let $\mathcal{CR} = (CR, \circ)$ represent the cross ratio group introduced in Problem 2 of Investigation 5 and consider the product group $\mathcal{CR} \times \mathcal{Z}_4$.

Part (a). What is the inverse of the element $(u, 3)$ in the product group?

Part (b). Construct the powers $(r, 2)^{-1}$, $(r, 2)^{-2}$, $(r, 2)^{-3}$, $(r, 2)^{-4}$, $(r, 2)^{-5}$, and $(r, 2)^{-6}$.

Homework.

Problem 1. Construct the operation table for the product group $\mathcal{Z}_3 \times \mathcal{Z}_4$. Is this group isomorphic to the group $\mathcal{Z}_{12} = (\mathbb{Z}_{12}, \boxplus_{12})$?

Problem 2. Construct the operation table for the product group $\mathcal{Z}_6 \times \mathcal{Z}_2$. Is this group isomorphic to the group $\mathcal{Z}_{12} = (\mathbb{Z}_{12}, \boxplus_{12})$?

Problem 3. Consider the element $(2,3) \in \mathbb{Z}_3 \times \mathbb{Z}_4$.

Part (a). Compute the powers $(2,3)^1$, $(2,3)^2$, …, $(2,3)^{12}$ in the group $\mathcal{Z}_3 \times \mathcal{Z}_4$.

Part (b). Explain why $\mathcal{Z}_3 \times \mathcal{Z}_4$ must be isomorphic to the group $\mathcal{Z}_{12}$.

Problem 4. Explain why $\mathcal{Z}_6 \times \mathcal{Z}_2$ is *not* isomorphic to the group $\mathcal{Z}_{12}$.

Problem 5. What is the inverse of the element $(2,3,4)$ in the product group $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_5$?

Problem 6. What is $(2,3,4)^{-7}$ in the product group $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_5$?

Problem 7. We say that a group $\boldsymbol{\mathcal{X}} = (X,*)$ is *cyclic* provided $X = \text{Pow}[a]$ for some $a \in X$. We call a a *generator* for the group if this is the case. The groups $\boldsymbol{\mathcal{Z}}_n = (\mathbb{Z}_n, \boxplus_n)$ are all cyclic with generator $a = 1$.

Part (a). Is the group $\boldsymbol{\mathcal{Z}} = (\mathbb{Z}, +)$ of integers cyclic? Explain.

Part (b). Is the Klein Four-Group cyclic? Explain.

Part (c). Is the group $\boldsymbol{\mathcal{Z}}_6 \times \boldsymbol{\mathcal{Z}}_2$ cyclic?

Part (d). Is the group $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_4$ cyclic?

Problem 8. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ are groups, and suppose that $f : X \to Y$ is an isomorphism. Use the method of mathematical induction to prove that $f(a^n) = [f(a)]^n$ for every positive integer n.

Problem 9. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a finite group. Prove the following result.

- The group $\boldsymbol{\mathcal{X}} = (X,*)$ is cyclic if and only if it is isomorphic to $\boldsymbol{\mathcal{Z}}_n$ for some integer $n > 1$.