In the next few exercises, we will consider the group $\boldsymbol{S}_{\Delta}$ of triangle symmetries. Here is the operation table for this group.

$*$	$\boldsymbol{RRR}$	$\boldsymbol{RR}$	$\boldsymbol{R}$	$\boldsymbol{F}$	$\boldsymbol{FR}$	$\boldsymbol{F(RR)}$
$\boldsymbol{RRR}$	RRR	RR	R	F	FR	$F(RR)$
$\boldsymbol{RR}$	RR	R	RRR	FR	$F(RR)$	F
$\boldsymbol{R}$	R	RRR	RR	$F(RR)$	F	FR
$\boldsymbol{F}$	F	$F(RR)$	FR	RRR	R	RR
$\boldsymbol{FR}$	FR	F	$F(RR)$	RR	RRR	R
$\boldsymbol{F(RR)}$	$F(RR)$	FR	F	R	RR	RRR

Problem 1. Identify all of the nonempty subsets of $\{R, RR, RRR, F, FR, F(RR)\}$ that are closed under the $*$ operation.

Problem 2. Consider the subsets you identified in Problem 1. Are there any features that they all have in common?

Problem 3. Consider the group $\boldsymbol{Z}_{12} = (\mathbb{Z}_{12}, \boxplus_{12})$. Identify all of the nonempty subsets of $\mathbb{Z}_{12}$ that are closed under the operation.

Problem 4. Consider the subsets you identified in Problem 3. Are there any features that they all have in common?

> ***Subgroup of a Group***
>
> Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group. We say that a subset H of the set X is a *subgroup* of the group $\boldsymbol{\mathcal{X}}$ provided the set H is a group in its own right under the operation *. In other words,
>
> 1. The set H is closed under the binary rule *.
> 2. There is an identity element for the set H under the operation *.
> 3. Every member of H has an inverse (relative to the identity for H) under the operation *.

Problem 5. Are the following statements true or false? Explain your answer.

- Every nonempty subset of triangle symmetries that is closed under the $*$ operation is a subgroup of the group $\boldsymbol{\mathcal{S}}_\Delta$.
- Every nonempty subset of $\mathbb{Z}_{12}$ that is closed under the operation $\boxplus_{12}$ is a subgroup of the group $\boldsymbol{\mathcal{Z}}_{12}$.

Problem 6. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and let $a \in X$. Is the set $\mathrm{Pow}[a]$ a subgroup of $\boldsymbol{\mathcal{X}}$? Explain.

Problem 7. Consider the group $\boldsymbol{\mathcal{Z}} = (\mathbb{Z}, +)$ of integers under addition.

Part (a). Is the set $\mathbb{Z}^+$ of positive integers closed under addition?

Part (b). Is the set $\mathbb{Z}^+$ a subgroup of $\boldsymbol{\mathcal{Z}}$? Explain.

Problem 8. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and suppose that $H \subseteq X$ is a subgroup of $\boldsymbol{\mathcal{X}}$. Let ε be the identity for X under *, and let δ be the identity for H under *.

Part (a). Prove that we must have $\varepsilon = \delta$. (Hint: Let $a \in H$ and consider the equation $a * x = a$ in *both* groups.)

Part (b). Let $a \in H$, and suppose that a^{-1} represents the inverse for a in the group $\boldsymbol{\mathcal{X}}$. Prove that $a^{-1} \in H$ as well.

Homework.

Problem 1. There are ten subgroups of the group $\boldsymbol{\mathcal{S}}_{\times}$ of cross symmetries.

Part (a). Construct all ten of these subgroups.

Part (b). What is the *smallest* subgroup that contains both RR and F?

Part (c). What is the *smallest* subgroup that contains both R and F?

Problem 2. Consider the group $\boldsymbol{\mathcal{Z}}_6 \times \boldsymbol{\mathcal{Z}}_8$. What is the *smallest* subgroup of this group that contains both (3,4) and (0,6)?

Problem 3. Consider the group $\boldsymbol{\mathcal{Z}}_6 \times \boldsymbol{\mathcal{S}}_{\times}$. What is the *smallest* subgroup of this group that contains both $(3, F)$ and $(0, RR)$?

Problem 4. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and suppose that H is a nonempty subset of X. Furthermore, suppose that $a * b^{-1} \in H$ for all $a, b \in H$.

Part (a). Prove that the identity of $\boldsymbol{\mathcal{X}}$ is a member of H.

Part (b). Let $u \in H$. Use Part (a) to prove that $u^{-1} \in H$.

Part (c). Let $m, n \in H$. Use Part (b) to prove that $m * n \in H$.

Problem 2 is often called the *subgroup test* because it tells us that *if* we can show that a nonempty subset H of a group $\boldsymbol{\mathcal{X}}$ contains $a * b^{-1}$ for all $a, b \in H$, *then* we know that H is a subgroup of $\boldsymbol{\mathcal{X}}$.

Problem 5. Consider the general linear group $\boldsymbol{GL_2} = (M_2,*)$ of 2×2 invertible matrices under matrix multiplication. Use the subgroup test to show that the following set is a subgroup of $\boldsymbol{GL_2}$.

$$H = \{A \in U_{2\times 2} : \mathrm{Det}(A) = 1\}$$

You may need to review the properties of matrix determinants from linear algebra.

Problem 6. Prove that the group $\boldsymbol{\mathcal{Z}} = (\mathbb{Z}, +)$ is isomorphic to its subgroup $n\mathbb{Z}$ for any fixed positive integer n.

Problem 7. Consider the group $\boldsymbol{\mathcal{Z}} = (\mathbb{Z}, +)$ along with any subgroup H of $\mathbb{Z}$ that contains at least two elements.

Part (a). Explain why H must contain a positive integer.

Part (b). Let n be the *smallest* positive member of H. Explain why $n\mathbb{Z} \subseteq H$.

Part (c). Let $a \in H$. Use the Division Algorithm and Part (b) to explain why we must have $a = nk$ for some integer k. (Therefore, we know $H = n\mathbb{Z}$.)

A group $\boldsymbol{\mathcal{X}} = (X,*)$ is *cyclic* provided $X = \mathrm{Pow}[a]$ for some $a \in X$. Problem 7 tells us that the subsets $n\mathbb{Z}$ are the *only* subgroups of the group $\boldsymbol{\mathcal{Z}}$. In particular, every subgroup of $\boldsymbol{\mathcal{Z}}$ is cyclic.