Let's start with some basic terminology. A *quantity* is any attribute of an object or process that can be *quantified* (measured). When we identify a quantity, there are always three questions that must be considered:

1. Precisely what is the quantity?
2. What unit of measure will be used to quantify it?
3. What is the reference point for the measurement?

A quantity may be *constant* (which means the measurement does not change during the process under consideration) or it may be *varying* (which means the measurement does change). We typically assign a *variable name* to represent the changing measure values of a varying quantity.

Example 1. Suppose you place an empty glass under water faucet and turn on the water so that the glass begins to fill. What are some constant quantities associated with this process?

First, note that "the glass" is *not* a quantity in this process, since there is no way to measure the "glass-ness" of the glass. Likewise, "the water" is not a quantity. *Aspects* or *attributes* of "the glass" or "the water" could all be quantities we would want to consider.

- The dimensions of the glass would all be constant quantities, since we are only considering a single glass in this process.
- The *total* time the water faucet is left on would be a constant quantity, since the glass is only being filled once.
- The *total* volume of water in the glass would be a constant quantity.

Notice that we did not properly identify any of the constant quantities above. For example, to properly identify the constant quantity "total time the water faucet is left on" we would need to be more specific.

- "The total time the water faucet is left on", measured in minutes since the faucet was first turned on, is a constant quantity.

This time around, we were careful to answer the three questions.

1. Precisely what is the quantity? *The total time the water faucet is left on*
2. What unit of measure will be used to quantify it? *Minutes*
3. What is the reference point for the measurement? *The moment the faucet was first turned on*

Here is another properly defined constant quantity associated with the process:

- "The height of the glass", measured in inches, is a constant quantity.

Notice that this time, we answered Questions 1 and 2, but did not try to answer Question 3. (We could have answered this question by stating that the height would be measured from the base of the glass.) Question 3 is most important when we are defining quantities that involve time or distance traveled; nonetheless, we should always consider Question 3 when defining any quantity.

Problem 1. Identify some *varying* quantities in the glass-filling process of Example 1. Be careful to properly define each quantity.

Problem 2. Stan begins eating a bowl of soup at 12:00 noon.

Part (a). Properly identify at least two constant quantities associated with this process.

Part (b). Properly identify at least two varying quantities associated with this process.

We typically assign a *variable name* to represent the values of a varying quantity associated with a process. A variable name for the values of a varying quantity can be anything you choose, but it is usually a single letter that reflects the name of the quantity.

Example 2. Laurel starts inflating a spherical balloon. What are some varying quantities associated with this process, and how are the values of these quantities related?

Solution. To answer this question, it will be helpful to both identify varying quantities and assign variable names to represent the possible values of these quantities. This can be done in one step for each quantity. For example,

- Let R represent the values of the quantity “the radius of the balloon,” measured in centimeters.
- Let V represent the values of the quantity “the volume of the balloon,” measured in cubic centimeters.
- Let S represent the values of the quantity “the surface area of the balloon,” measured in square centimeters.
- Let t represent the values of the quantity “the time passed since the balloon began to inflate,” measured in seconds since the balloon began to inflate.

The volume, radius, and surface area of the balloon are all related by standard formulas. In particular, we know

$$S = f(R) = 4\pi R^2 \qquad \text{and} \qquad V = g(R) = \frac{4\pi R^3}{3}$$

We also know that the values of S, V, and R all depend in some way on the values of t, since these quantities are changing with time. However, we are not given enough information to know what this relationship is.

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Covarying Quantities

We say that two or more quantities *covary* provided changes in the measure of one or more quantities cause changes in the measure of the other quantities.

If we can construct a formula that gives the measures of one quantity as a function of the measures of the other covarying quantities, then we say this quantity is *explicitly* related to those quantities. If we cannot construct such a formula, then we say that the quantity is *implicitly* related to those quantities.

In the previous example, the volume and radius quantities, and the surface area and radius quantities are explicitly related, since we can construct formulas that give the measure of surface area and volume in terms of the measure of the radius. However, the volume, surface, and radius quantities are all implicitly related to the time quantity, since we do not have enough information to construct formulas that give the measure of volume, surface area, or radius in terms of the measure of time.

Problem 2. In the previous example, are the surface area and volume quantities implicitly or explicitly related? Explain.

Problem 3. Suppose that a drop of oil is placed on the surface of a pool. The layer of oil on the surface spreads in the form of a circular disc.

Part (a). Correctly identify at least four varying quantities in this process.

Part (b). Identify pairs of covarying quantities for this process. Which pairs of covarying quantities are explictly related?

Problem 4. Naomi turns on an LED sign which shows a shaded rectangle whose dimensions can change.

Part (a). Correctly identify at least four varying quantities in this process.

Part (b). Identify four sets of two or three of your quantities that covary.

Part (c). Which of your covarying quantities from Part (b) are implicitly related?

Part (d). For each quantity that is explicitly related to one or more other covaring quantities, construct a formula that gives the measure of that quantity as a function of the measures of the others.

Homework.

Problem 1. Natasha's bathtub drain is clogged. She pours some Draino into the drain, and water starts to drain out of the tub. Explain why the following do not represent quantities in the process of the tub draining.

(a) the tub (b) the water (c) Natasha

Problem 2. Each of the following *incorrectly* identifies a varying quantity in the process of draining Natasha's bathtub. Rewrite each statement so that it correctly identifies a varying quantity.

(a) the temperature of the water

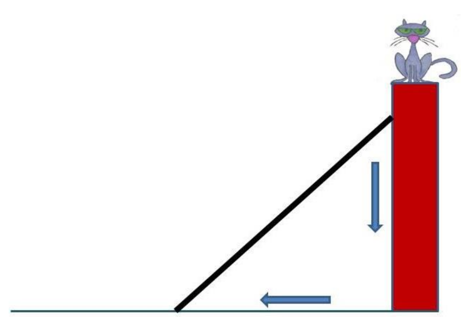
(b) the time passed

Problem 3. Justo takes a glass containing twelve ounces of water and pours it into the water bowl for his cat, Mr. Hipsickles. Correctly identify as many covarying quantities for this process as you can.

Problem 4. Justo has a twelve-foot ladder leaning against a vertical wall as shown below. Mr. Hipsickles scampers up the ladder, causing the the base of the ladder to slide away from the wall.

Part (a). Correctly identify as many covarying quantities associated with this process as you can. Which of the covarying quantities are implicitly related?

Part (b). For each quantity that is explicitly related to one or more other covaring quantities, construct a formula that gives the measure of that quantity as a function of the measures of the others. Be careful to identify the variables you use. (Area formulas and the Pythagorean Theorem are helpful.)



Answers to the Homework.

Problem 1. In each case, none of the items represent an *attribute* of the process that can be *measured*. Various attributes of the water (like temperature, volume, etc.) or the tub (like volume, mass, weight, etc.) could serve as quantities.

Problem 2. The temperature of the water, measured in degrees Celsius, would be a quantity in the process. This quantity could vary or could be constant. We don't have enough information to know.

The time passed since the tub began to drain, measured in seconds, would be a varying quantity in the process.

Problem 3. First, here is a list of varying quantities.

1. The time passed since Justo began pouring water from the glass, measured in seconds.
2. The weight of water in the glass, measured in ounces.
3. The weight of water in the bowl, measured in ounces.
4. The volume of water in the glass, measured in cubic inches.
5. The volume of water in the bowl, measured in cubic inches.

All of these quantities covary with each other. For example, as the measures of the time quantity increase, the measures of the weight of water in the glass will decrease. There could be other covarying quantities. For example, the height of the glass above the bowl, measured in inches, *might* be a varying quantity. However, there is not enough information to know for certain. Typically, we only concern ourselves with quantities that we are certain vary or remain constant.

Problem 4. Here is a list of varying quantities, along with variable names to represent their measurement values.

1. The area A of the triangle formed by the ladder, the wall, and the ground, measured in square feet.
2. The vertical distance y between the top of the ladder and the ground, measured in feet.
3. The horizontal distance x between the base of the ladder and the wall, measured in feet.
4. The time t passed, measured in seconds, since the ladder began to slide down the wall.

All of these quantities covary with each other, and some of the covariances are implicit. In particular, the values of A covary implicitly with the values of t. Likewise, the values of y and the values of x covary implicitly with the values of t.

On the other hand, the values of A covary explicitly with the values of x and y. In particular, we know

$$A = \frac{1}{2}(xy)$$

By solving this equation for x or for y, we can create additional functions. Indeed, we know that

$$y = 2\left(\frac{A}{x}\right) \quad x = 2\left(\frac{A}{y}\right)$$

Thus, we also know that the values of x covary explicitly with the values of A and y. Likewise, we know that the values of y covary explicitly with the values of A and x.

Finally, we also know that the values of x and y covary explicitly with each other. In particular, since the values of x and y represent distances, we can assume that $x \geq 0$ and $y \geq 0$. Therefore, the Pythagorean Theorem tells us we have

$$x = \sqrt{144 - y^2} \qquad y = \sqrt{144 - x^2}$$

It is important to note that the Pythagorean Theorem really tells us that $x^2 + y^2 = 144$. Therefore, if it were *not* reasonable to assume that $x \geq 0$ and $y \geq 0$, the covariance between x and y would be *implicit*. This is because, if we *did not know* the values of x and y are nonnegative, we could only write

$$x = \pm\sqrt{144 - y^2} \qquad y = \pm\sqrt{144 - x^2}$$

There are *two* formulas that express the values of x in terms of the values of y; and likewise, there are *two* formulas that express the values of y in terms of the values of x. Consequently, these covariance between x and y would be implicit.