Suppose that $y = f(x)$ is a function. If the function f is locally linear at the point $(a, f(a))$, then it is common to say that the function f is *differentiable* at the input value $x = a$. The various processes we use to determine the derivative function for f are known collectively as *differentiation*.

Differentiating a function $y = f(x)$ amounts to determining the derivative function $r = f'(x)$. To do this, we must either

(1) Construct the graph of the derivative function f' from the formula or graph for the function f
(2) Find a way to determine a formula for the derivative function f' directly from its limit definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Determining a formula from the limit definition requires us to show that the average rate of change function

$$g_x(h) = \frac{f(x+h) - f(x)}{h}$$

has a removable discontinuity at the input value $h = 0$; and if g_x is an algebraic function, this can usually be done using basic algebra.

Problem 1. Consider the constant function $y = f(x) = 5$.

Part (a). What is the average rate of change for this function on the input interval from $x = 1$ to $x = 10$?

Part (b). Construct the function $r = g_a(h)$ that gives the average rate of change for the function f on the input interval from $x = a$ to $x = a + h$.

Part (c). Based on your answer to Part (b), what do you think the value of $f'(a)$ should be?

Problem 2. Suppose that K is any real number, and consider the function $y = f(x) = K$.

Part (a). What do you think the formula for the derivative function $r = f'(x)$ should be?

Part (b). Use the limit definition of the derivative to prove your formula is correct.

Example 1. Differentiate the function $y = f(t) = t^{-1}$ with respect to the input variable t.

Solution. For any fixed value of the input variable t, consider the average rate of change function

$$g_t(h) = \frac{f(t+h) - f(t)}{h} = \left(\frac{1}{h}\right)\left(\frac{1}{t+h} - \frac{1}{t}\right)$$

We want to use algebra to simplify the rightmost formula. The goal of the simplification is to show that the factor $1/h$ can be cancelled from the formula. Observe

$$\begin{aligned} g_x(h) &= \left(\frac{1}{h}\right)\left(\frac{1}{t+h} - \frac{1}{t}\right) \\ &= \left(\frac{1}{h}\right)\left(\frac{1}{t+h}\left[\frac{t}{t}\right] - \frac{1}{t}\left[\frac{t+h}{t+h}\right]\right) \\ &= \left(\frac{1}{h}\right)\left(\frac{t-(t+h)}{t(t+h)}\right) \\ &= \left(\frac{1}{h}\right)\left(-\frac{h}{t(t+h)}\right) \\ &= -\frac{1}{t(t+h)} \qquad\qquad (h \neq 0) \end{aligned}$$

Now, having shown that the function g_t has a removable discontinuity at the input value $h = 0$, we can determine the formula for the derivative function for f. Observe

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \frac{-1}{t(t+h)} = -\frac{1}{t^2}$$

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The derivative function for a function $y = f(x)$ with respect to the input variable x is often denoted by the operation symbol

$$\frac{d}{dx}[\,f(x)\,]$$

This symbol means "the derivative function with respect to *x* of the function appearing in brackets." For example, we can now write

$$\frac{d}{dt}[t^{-1}] = -\frac{1}{t^2}$$

It is worth pointing out that the derivative formula is *invariant* regarding how we choose to name the input variable. For example, it will also be true that

$$\frac{d}{dx}[x^{-1}] = -\frac{1}{x^2} \qquad \text{and} \qquad \frac{d}{da}[a^{-1}] = -\frac{1}{a^2}$$

Problem 3. Use the procedure from Example 1 to determine the formula for $f'(z) = \frac{d}{dz}[z^2]$.

Problem 4. Differentiate the function $y = f(s) = s^3$ following the method used in Example 1. Hint: Remember that $(s + h)^3 = s^3 + 3s^2h + 3sh^2 + h^3$.

Example 2. Differentiate the function $b = f(w) = w^{1/2}$ with respect to the input variable w.

Solution. Once again, consider the average rate of change function

$$g_w(h) = \frac{f(w+h) - f(w)}{h} = \frac{(w+h)^{1/2} - w^{1/2}}{h}$$

Proving that this function has a removable discontinuity at the input value $h = 0$ requires a bit more ingenuity than we needed in Example 1, because we cannot "expand" a binomial raised to a fractional power the same way we can expand a binomial raised to an integer power.

The key to simplifying in this case is to make the observation that, thanks to the laws of exponents, we know for any expressions A and B, the following equation is true as long as $A^{1/2}$ and $B^{1/2}$ are defined:

$$\left(A^{1/2} - B^{1/2}\right)\left(A^{1/2} + B^{1/2}\right) = A - B$$

With this in mind, observe that

$$g_w(h) = \frac{(w+h)^{1/2} - w^{1/2}}{h}$$

$$= \left[\frac{(w+h)^{1/2} - w^{1/2}}{h}\right]\left[\frac{(w+h)^{1/2} + w^{1/2}}{(w+h)^{1/2} + w^{1/2}}\right]$$

$$= \left(\frac{1}{h}\right)\left[\frac{(w+h) - w}{(w+h)^{1/2} + w^{1/2}}\right]$$

$$= \left(\frac{1}{h}\right)\left[\frac{h}{(w+h)^{1/2} + w^{1/2}}\right]$$

$$= \frac{1}{(w+h)^{1/2} + w^{1/2}} \qquad (h \neq 0)$$

Now that we have demonstrated that the function g_w has a removable discontinuity at the input value $h = 0$, we can determine for the formula for the derivative function for $f(w) = w^{1/2}$. Observe

$$f'(w) = \frac{d}{dw}\left[w^{1/2}\right] = \lim_{h\to 0}\frac{1}{(w+h)^{1/2} + w^{1/2}} = \frac{1}{w^{1/2} + w^{1/2}} = \frac{1}{2w^{1/2}}$$

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The algebra appearing in the previous examples and problems is daunting. However, there is a surprising pattern that appears in the formulas for the derivative functions we have constructed. Observe

$$f(x) = x^{-1} \quad \Rightarrow \quad f'(x) = -\frac{1}{x^2} = (-1)x^{-2} = (-1)x^{-1-1}$$

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x = (2)x^1 = (2)x^{2-1}$$

$$f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2 = (3)x^{3-1}$$

$$f(x) = x^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2x^{1/2}} = \left(\frac{1}{2}\right)x^{-1/2} = \left(\frac{1}{2}\right)x^{1/2-1}$$

Power Rule for Differentiation

If q is any rational number, then the derivative function for the function $y = f(x) = x^q$ is the function defined by

$$r = f'(x) = qx^{q-1}$$

Problem 5. Use the Power Rule to help you construct the point-slope formula for the line tangent to the graph of the function $y = x^{-4}$ at the point $(2, f(2))$.

Problem 6. Consider the function $y = f(x) = x^{3/2}$. Are there any values of x where $f'(x) = 6$?

The Power Rule for Differentiation is an example of a *specific derivative formula*. The Power Rule summarizes a pattern seen whenever we differentiate a power function. Here is a specific derivative formula that you proved in Problems1 and 2.

Constant Function Rule for Differentiation

If $y = f(x)$ is any constant function, then the derivative function for f is the function defined by $r = f'(x) = 0$.

It is worth noting that the *converse* of this Specific Formula is also valid. In other words,

If $y = f(x)$ is a differentiable function, and $r = f'(x) = 0$ for all input values x, then the function f must have constant output.

Proof.

Verifying the validity of this statement requires the Mean Value Theorem. We want to show that the function f has constant output. This means we want to show that for *any* distinct input values $x = a$ and $x = b$, we have $f(a) = f(b)$. To this end, let $x = a$ and $x = b$ be distinct input values. The Mean Value Theorem tells us there is some input value $a < c < b$ such that

$$0 = f'(c) = \frac{f(b) - f(a)}{b - a}$$

It follows that $0 = f(b) - f(a)$; and we must conclude that $f(a) = f(b)$, as desired.

The converse of the Constant Rule forms the foundation for the branch of mathematics called *differential equations* and is key to much of the work you will do in Calculus II.

Now, let's consider the *exponential* functions. An exponential function has the form $y = f(x) = B^x$, where B is a positive constant. Exponential functions are transcendental functions; consequently, working with the average rate of change function for an exponential function will pose special challenges. Consider the average rate of change function

$$g_x(h) = \frac{f(x+h) - f(x)}{h} = \frac{B^{x+h} - B^x}{h}$$

Once again, the goal is to prove that this function has a removable discontinuity at $h = 0$. However, this time basic algebra will be of limited use. The input variable h in the numerator is trapped in the exponent, and no amount of algebra will be able to undo this fact. Let's see what basic algebra is able to tell us.

Problem 7. The Laws of Exponents tell us that $B^{x+h} = B^x \cdot B^h$. Use this fact to show that

$$\frac{d}{dx}[B^x] = \lim_{h \to 0} \frac{B^{x+h} - B^x}{h} = B^x \cdot \lim_{h \to 0} \frac{B^h - 1}{h}$$

There are ways to determine the exact value of this limit process; however, these methods are beyond the scope of this course. Therefore, instead of tackling the limit process directly, let's look at estimates of this limit process for various values of B and see if there is a pattern.

Problem 8. For example, let's consider the base $B = 2$.

Part (a). To an accuracy of at least three decimal places, what is the approximate value of

$$\lim_{h \to 0} \frac{2^h - 1}{h}$$

Part (b). Use your answer to Part (a) to complete the following formula.

$$\frac{d}{dx}[2^x] \approx \underline{\hspace{3cm}} \cdot 2^x$$

Problem 9. For each specified value of B in the table below, approximate the value of

$$\lim_{h \to 0} \frac{B^h - 1}{h}$$

Make sure your estimate is accurate to at least three decimal places.

Specified Value of *B*	0.111	0.20	0.25	0.333	3.0	4.0	5.0	9.0
Approximate Value of $\lim_{h \to 0} \frac{B^h - 1}{h}$								

Problem 10. Use your information from Problem 9 to complete the following formulas.

$\frac{d}{dx}[(0.111)^x] \approx$ _________ $\cdot (0.111)^x$ $\frac{d}{dx}[(0.20)^x] \approx$ _________ $\cdot (0.20)^x$

$\frac{d}{dx}[(0.25)^x] \approx$ _________ $\cdot (0.25)^x$ $\frac{d}{dx}[(0.333)^x] \approx$ _________ $\cdot (0.333)^x$

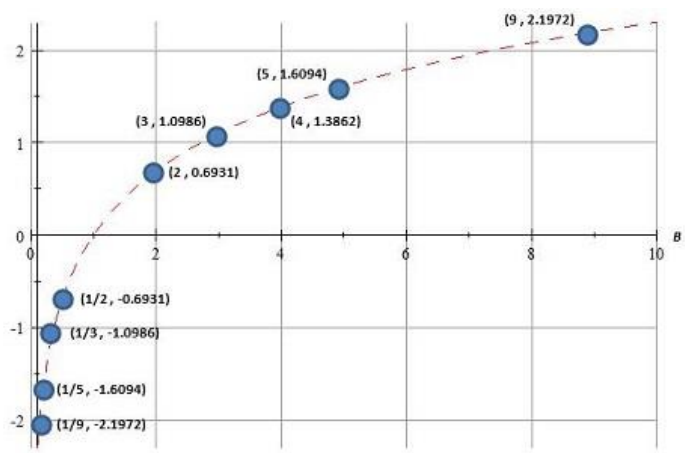
$\frac{d}{dx}[(3.0)^x] \approx$ _________ $\cdot (3.0)^x$ $\frac{d}{dx}[(4.0)^x] \approx$ _________ $\cdot (4.0)^x$

$\frac{d}{dx}[(5.0)^x] \approx$ _________ $\cdot (5.0)^x$ $\frac{d}{dx}[(5.0)^x] \approx$ _________ $\cdot (5.0)^x$

Simply looking at the approximations to these limiting processes does not help us identify a pattern relating all of the values. However, that changes when we plot the ordered pairs

$$\left(B, \lim_{h \to 0} \frac{B^h - 1}{h}\right)$$

for the various values of B we have considered.



These ordered pairs all lie on the curve that defines the function $y = \ln(B)$. This observation provides strong evidence for the following specific derivative formula.

Exponential Function Rule for Derivatives

If B is a positive constant, then the derivative function for the function $y = f(x) = B^x$ is defined by the formula

$$r = f'(x) = B^x \cdot \ln(B)$$

Problem 11. What is the point-slope formula for the line tangent to the graph of $y = f(x) = \left(\frac{1}{3}\right)^x$ at the point $(2, f(2))$?

Example 3. Differentiate the function $y = f(a) = a^2 + a^{-1}$.

Solution. Consider the average rate of change function

$$\begin{aligned} g_a(h) &= \frac{f(a+h) - f(a)}{h} \\ &= \left(\frac{1}{h}\right)\left(\left[(a+h)^2 + \frac{1}{a+h}\right] - \left[a^2 + \frac{1}{a}\right]\right) \\ &= \left(\frac{1}{h}\right)\left([(a+h)^2 - a^2] + \left[\frac{1}{a+h} - \frac{1}{a}\right]\right) \\ &= \left(\frac{1}{h}\right)[(a+h)^2 - a^2] + \left(\frac{1}{h}\right)\left[\frac{1}{a+h} - \frac{1}{a}\right] \end{aligned}$$

By simply rearranging the terms in the formula for the function g_a we have recast the function as the sum of two average rate of change functions that we have already worked with. In particular, we can see that

$$\begin{aligned} \frac{d}{da}[a^2 + a^{-1}] &= \lim_{h\to 0}\left\{\left(\frac{1}{h}\right)[(a+h)^2 - a^2] + \left(\frac{1}{h}\right)\left[\frac{1}{a+h} - \frac{1}{a}\right]\right\} \\ &= \lim_{h\to 0}\left(\frac{1}{h}\right)[(a+h)^2 - a^2] + \lim_{h\to 0}\left(\frac{1}{h}\right)\left[\frac{1}{a+h} - \frac{1}{a}\right] \\ &= \frac{d}{da}[a^2] + \frac{d}{da}[a^{-1}] \\ &= 2a - \frac{1}{a^2} \end{aligned}$$

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In the previous example, we were tasked with constructing the derivative function for a function f whose formula was the sum of two functions whose derivatives we have already determined; and it turned out that the derivative function for f is simply the sum of the derivative formulas for the two component functions. This is a special example of the following rule.

Sum of Functions Rule for Derivatives

If $y = f(x)$ and $y = g(x)$ are differentiable functions, then the sum of these functions is also differentiable. In fact,

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Notice that this rule does not provide the formula for the derivative function of a *specific* function. Instead, it tells us how to determine the derivative for a sum of functions if we know the derivative for each function individually. The Sum Rule is an example of a *general derivative rule*.

Constant Multiple Rule for Derivatives

If $y = f(x)$ is a differentiable function, and if C is any constant, then the constant-multiple function defined by $y = g(x) = C \cdot f(x)$ is also differentiable. In fact,

$$\frac{d}{dx}[C \cdot f(x)] = C \cdot \frac{d}{dx}[f(x)]$$

The Constant Multiple Rule is another general rule for derivatives. To see why this general rule should be true, consider the average rate of change function

$$A_x(h) = \frac{g(x+h) - g(x)}{h} = \frac{C \cdot f(x+h) - C \cdot f(x)}{h} = C \cdot \frac{f(x+h) - f(x)}{h}$$

Since the function f is assumed to be differentiable, we may conclude

$$\frac{d}{dx}[g(x)] = C \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = C \cdot \frac{d}{dx}[f(x)]$$

Example 4 . What is the formula for the derivative function of $= f(x) = 3\sqrt{x} + 5 \cdot 4^x + 8$?

Solution. Observe that

$$\frac{d}{dx}\left[3\sqrt{x} + 5 \cdot 4^x + 8\right] = \frac{d}{dx}[3\sqrt{x}] + \frac{d}{dx}[5 \cdot 4^x] + \frac{d}{dx}[8] \qquad \text{Sum Rule for Derivatives}$$

$$= 3\frac{d}{dx}[\sqrt{x}] + 5 \cdot \frac{d}{dx}[4^x] + \frac{d}{dx}[8] \qquad \text{Constant Multiple Rule for Derivatives}$$

$$= \frac{3}{2\sqrt{x}} + 4^x \ln(4) + 0 \qquad \text{Power Rule, Exponential Rule, \& Constant Function Rule}$$

(Note that we took for granted the fact that $\sqrt{x} = x^{1/2}$.)

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Problem 12. If $y = f(t) = 2t^{2/3} - 6 \cdot e^t$, then what is the value of $f'(3)$? Remember, the symbol *e* represents *euler's constant*. ($e \approx 2.7183$)

Problem 13. Determine the input values where the tangent line to the graph of $y = f(x) = 3x^3 - 7x$ is horizontal.

Problem 14. Determine the input values where the instantaneous rate of change for the function defined by $y = f(x) = 3x - \sqrt{x}$ is equal to 1.

Homework.

Use the sum and constant multiple rules, along with the specific derivative formulas to differentiate the following functions.

(1) $f(x) = e^5$ **(2)** $g(z) = 5.2z + 2.3$ **(3)** $h(t) = 2t^3 - 3t^2 - 4t + \sqrt{3}$

(4) $f(x) = \frac{2x^{-3/4}}{5}$ **(5)** $g(z) = z^{5/3} - z^{2/3}$ **(6)** $h(t) = \sqrt[5]{t^2} - \frac{2}{t} + \pi$

(7) $f(x) = 4 \cdot 5^x + 2\sqrt{5x}$ **(8)** $g(z) = \sqrt{3}e^z + \frac{\pi}{\sqrt{z}}$ **(9)** $h(t) = 1 + 3^{t+2}$

Problem 10. Construct the formula for the line tangent to the graph of $y = f(x) = 2e^x + x$ at the point $(0, f(0))$.

Problem 11. Determine all of the input values where the function $y = f(x) = 4x^3 + 15x^2 - 18x + 1$ has a horizontal tangent line.

Problem 12. Determine all of the input values where the tangent line to the graph of the function $y = f(x) = \frac{2}{x} - 3x + 8$ is parallel to the line $y = g(x) = -4x + 5$.

Problem 13. Consider the function $y = f(x) = 4x - x^{-1}$. This function is continuous on the closed input interval $1 \le x \le 4$ and is differentiable on the open interval interval $1 < x < 4$. The Mean Value Theorem therefore guarantees there is at least one input value $1 < c < 4$ where the instantaneous rate of change for f is equal to the average rate of change for f on the interval from $x = 1$ to $x = 4$. Determine these input values.

Answers to the Homework.

(1) $f'(x) = 0$

(2) $g'(z) = 5.2$

(3) $h'(t) = 6t^2 - 6t - 4$

(4) $f'(x) = -\left(\frac{3}{10}\right)x^{-7/4}$

(5) $g'(z) = \frac{5z^{2/3} - 2z^{-1/3}}{3}$

(6) $h'(t) = \frac{2}{5}t^{-3/5} + \frac{2}{t^2}$

(7) $f'(x) = 4 \cdot 5^x \ln(5) + \sqrt{\frac{5}{x}}$

(8) $g'(z) = \sqrt{3}e^z - \frac{\pi}{2}z^{-3/2}$

(9) $h'(t) = 9 \cdot 3^t \ln(3)$

Problem 10. First, note that $f(0) = 2$ and $f'(x) = 2e^x + 1$. Consequently, $f'(0) = 3$. The formula for the tangent line will be $y = T(x) = 3x + 2$.

Problem 11. First, note that $f'(x) = 12x^2 + 30x - 18$. Now,

$$f'(x) = 0 \quad \Longrightarrow \quad 2x^2 + 5x - 3 = 0 \quad \Longrightarrow (2x - 1)(x + 3) = 0$$

Thus, the tangent line to the graph of the function f will be horizontal when $x = -3$ and when $x = 1/2$.

Problem 12. First, note that we want the slope of the tangent line to be $m = -4$. Now, observe that

$$f'(x) = -\left(3 + \frac{2}{x^2}\right)$$

$$f'(x) = -4 \quad \Longrightarrow \quad \left(3 + \frac{2}{x^2}\right) = 4 \quad \Longrightarrow 2 = x^2$$

Therefore, the tangent line will be parallel to the line $y = g(x) = -4x + 5$ at the input values $x = \pm\sqrt{2}$.

Problem 13. First, note that the average value for the function f on this input interval is

$$A = \frac{f(4) - f(1)}{4 - 1} = \frac{17}{4}$$

Also, we know that $f'(x) = 4 + x^{-2}$. Now,

$$f'(x) = \frac{17}{4} \quad \Longrightarrow \quad 4 + \frac{1}{x^2} = \frac{17}{4} \quad \Longrightarrow \quad 16x^2 + 4 = 17x^2 \quad \Longrightarrow \quad 4 = x^2 \quad \Longrightarrow \quad x = \pm 2$$

Only the input value $x = 2$ lies in the specified interval.