In this discussion, we will explore general derivative rules that tell us how to differentiate products and quotients of differentiable functions. We already have a general rule that tells us how to differentiate a *sum* of differentiable functions --- the derivative of a sum of functions is the sum of the derivatives for those functions.

Does the same process work for *products* of differentiable functions? In other words, is it true that the derivative of a product of differentiable functions is simply the product of their derivatives? We can use the graphing calculator to explore this possibility.

Graphing Derivative Functions Using the Graphing Calculator

Here are the steps for setting up the graphing calculator to sketch the graph of the derivative function for $y = f(x)$.

Step 1. Input the formula for the function f as Y1 in your calculator.
Step 2. Turn off the function Y1. (Use the left arrow key to move the cursor over the equal sign and press enter.)
Step 3. Go to the function Y2. Select MATH Option 8. (This is the procedure called "nDeriv".) The symbol below will appear.

$$\frac{d}{d[\ \ ]}(\ \ )\Big|_{[\ \ ]=}$$

The cursor will be flashing in the first box.
Step 4. Fill in the boxes so that you have

$$\frac{d}{d[x]}(Y1)\Big|_{[x]=x}$$

To place Y1 in the box, use the sequence VARS → YVARS → Function → Y1.

Problem 1. Consider the functions $y = f(x) = x^2e^x$, $y = g(x) = x^2$, and $y = h(x) = e^x$.

Part (a). Use your graphing calculator to sketch the graph of $r = f'(x)$ in the viewing window $-2 \leq x \leq 1$ and $-2 \leq y \leq 10$.

Part (b). Now use your graphing calculator to sketch the graph of the product $y = g'(x) \cdot h'(x)$ along with the graph of f'. Are the graphs of these two functions the same?

Problem 2. Let $y = f(x) = x^{12}$, and let $y = u(x) = x^4$ and $y = v(x) = x^8$.

Part (a): Use the power rule to differentiate the functions f, u, and v.

$\frac{d}{dx}[x^{12}] =$ $\qquad$ $\frac{d}{dx}[x^4] =$ $\qquad$ $\frac{d}{dx}[x^8] =$

Part (b): The laws of exponents tell us that $f(x) = u(x) \cdot v(x)$; consequently, there ought to be a relationship between $f'(x)$ and the functions $u'(x)$ and $v'(x)$. Show that it is possible to write the function f' as a combination of the four functions u, v, u', and v'.

$$\frac{d}{dx}[f(x)] =$$

Problem 3. Now, suppose that $y = g(x)$ and $y = h(x)$ are both differentiable functions.

Part (a). Based on your work in Part (b) of Problem 2, construct a possible formula for the derivative of the product function $y = g(x)h(x)$.

$$\frac{d}{dx}[g(x)h(x)] =$$

Part (b). Use your graphing calculator to see if your proposed formula works. Use the functions from Problem 1.

Problem 4. Use the formula you devised in Problem 3 to construct the derivative function for the function $y = f(x) = \sqrt{x} \cdot 2^x$.

Problem 5. Use the formula you devised in Problem 3 to construct the derivative function for the function $y = f(x) = x^3 \cdot 5^x$.

> ***Product Rule for Derivatives***
>
> If $y = g(x)$ and $y = h(x)$ are differentiable functions, then the function $y = f(x) = g(x) \cdot h(x)$ is also differentiable, and
> $$f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

Example 1. Use the specific derivative formulas and the product rule to differentiate the function
$$y = 3x^{-3/4} \cdot 4^x$$

Solution. When using derivative rules to differentiate a function, the general strategy is always the same. You apply general rules to break up the formula for the function into combinations of derivatives of progressively simpler formulas, until you are faced with derivatives that are *exact matches* to the specific derivative rules.

The formula is a product of two functions

$$\frac{dy}{dx} = \frac{d}{dx}[\underbrace{3x^{-3/4}}_{u(x)} \cdot \underbrace{4^x}_{v(x)}]$$

$$= 3\,x^{-3/4} \cdot \frac{d}{dx}[4^x] + 4^x \cdot \frac{d}{dx}[3\,x^{-3/4}]$$ Apply Product Rule

$$= 3\,x^{-3/4} \cdot \frac{d}{dx}[4^x] + 4^x \cdot 3\frac{d}{dx}[\,x^{-3/4}]$$ Apply Constant Multiple Rule

$$= 3\,x^{-3/4} \cdot 4^x \ln(4) + 4^x \cdot 3\left(-\frac{3}{4}x^{-7/4}\right)$$ Apply Specific Formulas

$$= 3\,x^{-3/4} \cdot 4^x \ln(4) - \frac{9}{4}x^{-7/4} \cdot 4^x$$ Make easy simplifications

****

Problem 6. What is the instantaneous rate of change for the function $a = h(t) = 4t - t^2 e^t$ at the input value $t = 1$?

Problem 7. Construct the formula for the second derivative function of the function f if we know that $y = f(x) = x^2 e^x$.

Reciprocal Rule for Derivatives

Suppose that $y = f(x)$ is a differentiable function. As long as $f(x) \neq 0$, the reciprocal function $y = R(x) = \frac{1}{f(x)}$ is also differentiable, and

$$R'(x) = -\frac{f'(x)}{f(x) \cdot f(x)}$$

To see why this formula is valid, let's consider the average value function $y = g_a(h)$ that gives the average value of the function R on the input interval from $x = a$ to $x = a + h$. Observe

$$\begin{aligned} g_a(h) &= \frac{R(a+h) - R(a)}{h} \\ &= \left(\frac{1}{h}\right)\left(\frac{1}{f(a+h)} - \frac{1}{f(a)}\right) \\ &= \left(\frac{1}{h}\right)\left[\frac{f(a) - f(a+h)}{f(a+h)f(a)}\right] \\ &= -\left(\frac{1}{f(a+h)f(a)}\right)\left[\frac{f(a+h) - f(a)}{h}\right] \end{aligned}$$

Consequently, we know

$$\begin{aligned} R'(a) &= \lim_{h \to 0} \frac{R(a+h) - R(a)}{h} \\ &= -\left(\lim_{h \to 0} \frac{1}{f(a+h)f(a)}\right)\left(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\right) = -\left(\frac{1}{f(a) \cdot f(a)}\right) f'(a) \end{aligned}$$

Problem 8. Use the Reciprocal Rule to construct the derivative function for the function defined by $y = R(x) = \frac{1}{2-3x^2}$.

Problem 9. Use the Reciprocal Rule to construct the derivative function for $y = g(x) = 3^{-x}$.

Problem 10. Use the Product Rule and the Reciprocal Rule to construct the derivative function for

$$y = g(x) = \left(\frac{1}{3x-1}\right) \cdot (x-2)$$

Write your answer as a single fraction.

Problem 11. Suppose that $y = U(x)$ and $y = L(x)$ are both differentiable functions. Use the Product Rule and the Reciprocal Rule to show that

$$\frac{d}{dx}\left[\frac{U(x)}{L(x)}\right] = \frac{L(x)U'(x) - U(x)L'(x)}{L(x) \cdot L(x)}$$

The formula you verified in Problem 11 is called the *Quotient Rule* for derivatives. The Quotient Rule provides a shortcut for differentiating any ratio of differentiable functions; however, you can always use the Product Rule and Reciprocal Rule together to accomplish the same thing.

Quotient Rule for Derivatives

Suppose that $y = U(x)$ and $y = L(x)$ are differentiable functions. As long as $L(x) \neq 0$, the function

$$y = f(x) = \frac{U(x)}{L(x)}$$

is also differentiable, and

$$f'(x) = \frac{L(x) \cdot U'(x) - U(x) \cdot L'(x)}{L(x) \cdot L(x)}$$

Problem 12. Use the Quotient Rule to differentiate the function $s = y(t) = \frac{t^2}{1-t^2}$.

Example 2. Construct the formula for the derivative function for the function

$$p = q(r) = 4r - \frac{r}{1+r^2}$$

Solution. We will need to apply several general rules to construct this formula. Observe

This formula is a sum of two functions

$$\frac{dp}{dr} = \frac{d}{dr}\left[4r - \frac{r}{1+r^2}\right]$$

This formula is a quotient of two functions

$$= 4\frac{d}{dr}[r] - \frac{d}{dr}\left[\frac{r}{1+r^2}\right]$$

Apply Sum and Constant Multiple Rules

This formula is a sum of two functions

$$= 4\frac{d}{dr}[r] - \left[\frac{1}{(1+r^2)^2}\right]\left[(1+r^2)\frac{d}{dr}[r] - r\frac{d}{dr}[1+r^2]\right]$$

Apply Quotient Rule

$$= 4\frac{d}{dr}[r] - \left[\frac{1}{(1+r^2)^2}\right]\left[(1+r^2)\frac{d}{dr}[r] - r\left(\frac{d}{dr}[1] + \frac{d}{dr}[r^2]\right)\right]$$

Apply Sum Rule

$$= 4(1) - \left[\frac{1}{(1+r^2)^2}\right][(1+r^2)(1) - r(0+2r)]$$

Apply specific derivative formulas

$$= 4 - \frac{1-r^2}{(1+r^2)^2}$$

****

Homework.

Use the Product, Reciprocal, or Quotient Rules to construct the derivative functions for the following functions.

(1) $f(x) = (3x^2 - 5)e^x$

(2) $g(a) = \frac{1}{3+4x^4}$

(3) $h(w) = \left(w - \sqrt{w}\right)(1 - w^2)$

(4) $f(x) = \frac{\sqrt[3]{x}}{x-3}$

(5) $g(a) = \frac{1-a}{1+a}$

(6) $h(w) = \frac{1}{2^w - 4w}$

(7) $f(x) = \sqrt{x}(x^3 + 3x - 1)$

(8) $g(a) = \frac{3e^a \cdot 5^a}{2}$

(9) $h(w) = \frac{1}{w}(4^{-w+1})$

Problem 10. Construct the formula for $a = f''(x)$ for the function $y = f(x) = \frac{x^{5/3}e^x}{7}$.

Problem 11. Construct the formula for the line tangent to the graph of the function

$$y = f(x) = \frac{x}{2} - \frac{x}{x+1}$$

at the point $(1, f(1))$.

Answers to the Homework.

(1) $f'(x) = e^x(3x^2 + 6x - 5)$

(2) $g'(a) = -\frac{16x^3}{(3+4x^4)^2}$

(3) $h'(w) = \frac{(1-w^2)(2\sqrt{w}-1)}{2\sqrt{w}} - 2w(w - \sqrt{w})$

(4) $f'(x) = \frac{(x-3)x^{-2/3} - 3\sqrt[3]{x}}{3(x-3)^2}$

(5) $g'(a) = -\frac{2}{(1+a)^2}$

(6) $h'(w) = -\frac{2^w \ln(2) - 4}{(2^w - 4w)^2}$

(7) $f'(x) = \frac{x^3+3x-1}{2\sqrt{x}} + 3\sqrt{x}(x^2+1)$

(8) $g'(a) = \frac{3e^a \cdot 5^a(1+\ln(5))}{2}$

(9) $h'(w) = -\frac{4}{w \cdot 4^w}\left(\frac{1}{w} + \ln(4)\right)$

Problem 10. Observe that

$$f'(x) = \left(\frac{1}{7}\right)\frac{d}{dx}\left[x^{5/3}e^x\right] = \left(\frac{1}{7}\right)\left(\frac{d}{dx}\left[x^{5/3}\right]e^x + x^{5/3}\frac{d}{dx}\left[e^x\right]\right) = \left(\frac{e^x}{21}\right)\left(5x^{2/3} + 3x^{5/3}\right)$$

$$f''(x) = \left(\frac{1}{21}\right)\frac{d}{dx}\left[e^x\left(5x^{2/3} + 3x^{5/3}\right)\right]$$

$$= \left(\frac{1}{21}\right)\left(\frac{d}{dx}\left[e^x\right]\left(5x^{2/3} + 3x^{5/3}\right) + e^x\frac{d}{dx}\left[5x^{2/3} + 3x^{5/3}\right]\right)$$

$$= \left(\frac{e^x}{63}\right)\left(30x^{2/3} + 9x^{5/3} + 10x^{-1/3}\right)$$

Problem 11. Construct the formula for the line tangent to the graph of the function

$$y = f(x) = \frac{x}{2} - \frac{x}{x+1}$$

at the point $(1, f(1))$.

First, note that $f(1) = 0$. Now, observe that

$$f'(x) = \frac{1}{2} - \frac{1}{(x+1)^2} \qquad \Rightarrow \qquad f'(1) = \frac{1}{4}$$

The formula for the tangent line will therefore be $y = T(x) = \frac{1}{4}(x-1)$.