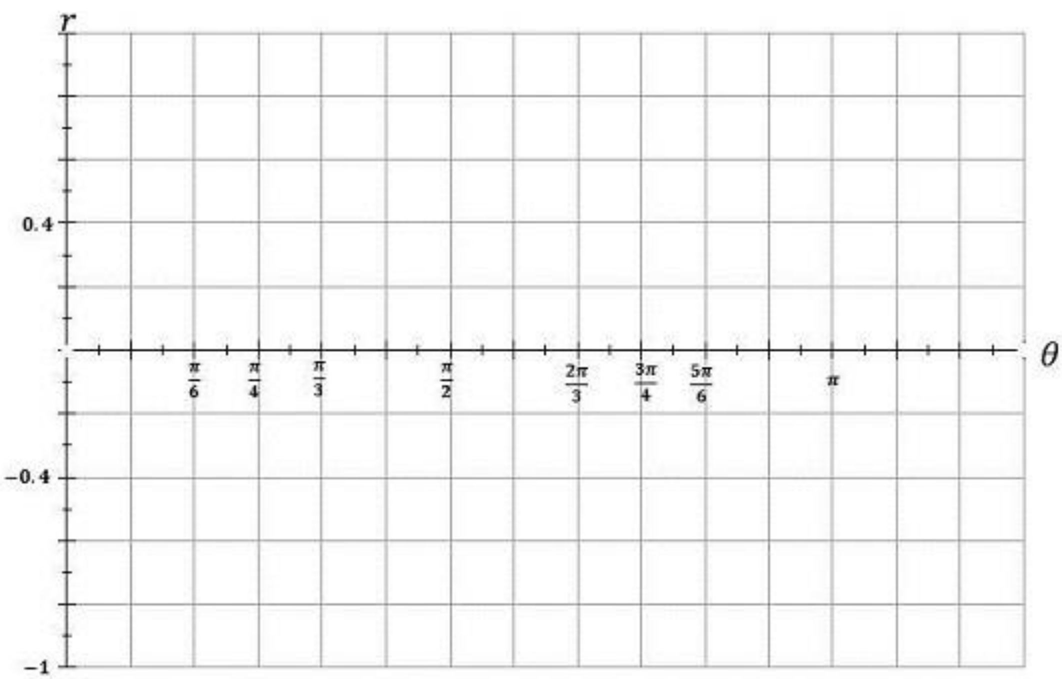
In this discussion, we will develop the special derivative formulas for the sine, cosine, and tangent functions. Like the exponential functions, the trigonometric functions are transcendental. Because of this fact, working with limits that involve the trigonometric functions pose special challenges.

Let's begin by sketching the graph of the derivative function for the function $y = f(\theta) = \sin(\theta)$. For any input value $\theta = a$, we know that

$$f'(a) \approx \frac{\sin(a + 0.0001) - \sin(a)}{0.0001}$$

Problem 1. Make sure your calculator is in radian mode and use this formula to fill in the table below. Then, use the data in this table to sketch the graph of the function $r = f'(\theta)$ on the grid provided.

Value of θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Approximate value of $f'(\theta)$									



Does the graph of the derivative function look familiar to you?

The sketch of the graph for the derivative function suggests that $f'(\theta) = \cos(\theta)$. We can give more evidence for this being the case by looking more closely at the average rate of change functions for $y = f(\theta) = \sin(\theta)$ and $x = j(\theta) = \cos(\theta)$. For any input value $\theta = a$, the average rate of change for the sine function and the cosine function on the input interval from $\theta = a$ to $\theta = a + h$ is given by

$$g_a(h) = \frac{f(a+h) - f(a)}{h} = \frac{\sin(a+h) - \sin(a)}{h}$$

$$G_a(h) = \frac{j(a+h) - j(a)}{h} = \frac{\cos(a+h) - \cos(a)}{h}$$

Both of these average rate of change functions have a discontinuity at the input value $h = 0$. The sine and cosine functions are transcendental; this means that algebra will be of little use in determining whether or not this discontinuity is removable. There is, however, a pair of trigonometric identities that can give insight into the nature of this discontinuity.

> ***Sum Identities for the Sine and Cosine Functions***
>
> If A and B are the radian measures of any two angles in standard position, then
>
> $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ $\qquad$ $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

Motivations and derivations for these identities can be found in any trigonometry text. If we apply these identities to the average rate of change functions above, we obtain the following variants on the formulas.

$$\begin{aligned} g_a(h) &= \frac{f(a+h) - f(a)}{h} \\ &= \frac{\sin(a+h) - \sin(a)}{h} \\ &= \frac{\sin(a)\cos(h) + \cos(a)\sin(h) - \sin(a)}{h} \\ &= \frac{[\sin(a)\cos(h) - \sin(a)] + \cos(a)\sin(h)}{h} \\ &= \frac{\sin(a)\cos(h) - \sin(a)}{h} + \frac{\cos(a)\sin(h)}{h} \\ &= \sin(a)\cdot\left[\frac{\cos(h) - 1}{h}\right] + \cos(a)\cdot\left[\frac{\sin(h)}{h}\right] \end{aligned}$$

$$\begin{aligned} G_a(h) &= \frac{j(a+h) - j(a)}{h} \\ &= \frac{\cos(a+h) - \cos(a)}{h} \\ &= \frac{\cos(a)\cos(h) - \sin(a)\sin(h) - \cos(a)}{h} \\ &= \frac{[\cos(a)\cos(h) - \cos(a)] - \sin(a)\sin(h)}{h} \\ &= \frac{\cos(a)\cos(h) - \cos(a)}{h} + \frac{\sin(a)\sin(h)}{h} \\ &= \cos(a)\cdot\left[\frac{\cos(h) - 1}{h}\right] - \sin(a)\cdot\left[\frac{\sin(h)}{h}\right] \end{aligned}$$

Problem 2. Use your graphing calculator to estimate the value of the following limits.

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} \qquad\qquad \lim_{h\to 0} \frac{\sin(h)}{h}$$

It is possible to determine the exact value of these limits; however, this determination is beyond the scope of this course. Based on your estimates, however, we can surmise that

$$\frac{d}{d\theta}[\sin(\theta)] = \lim_{h\to 0}\frac{\sin(\theta+h)-\sin(\theta)}{h}$$
$$= \sin(\theta)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(\theta)\lim_{h\to 0}\frac{\sin(h)}{h}$$
$$= \sin(\theta)\cdot \mathbf{0} + \cos(\theta)\cdot \mathbf{1}$$
$$= \cos(\theta)$$

$$\frac{d}{d\theta}[\cos(\theta)] = \lim_{h\to 0}\frac{\cos(\theta+h)-\cos(\theta)}{h}$$
$$= \cos(\theta)\lim_{h\to 0}\frac{\cos(h)-1}{h} - \sin(\theta)\lim_{h\to 0}\frac{\sin(h)}{h}$$
$$= \cos(\theta)\cdot \mathbf{0} - \sin(\theta)\cdot \mathbf{1}$$
$$= -\sin(\theta)$$

Problem 3. If $y = f(t) = t^2\sin(t)$, then what is the value of $f'(2)$?

Problem 4. Let $y = f(\theta) = [\sin(\theta)]^2$ and use the Product Rule to determine the formula for the derivative function $r = f'(\theta)$.

Pythagorean Identity

If θ is the radian measure of an angle in standard position, then the functions $x = f(\theta) = \cos(\theta)$ and $y = g(\theta) = \sin(\theta)$ are related by the following equation.

$$1 = [\cos(\theta)]^2 + [\sin(\theta)]^2$$

Problem 5. The *secant* and *tangent* functions are defined by the formulas

$$y = \sec(\theta) = \frac{1}{\cos(\theta)} \qquad m = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Part (a). Use the Reciprocal Rule to show that $\frac{dy}{d\theta} = \sec(\theta)\tan(\theta)$.

Part (b). Use the Quotient Rule and the Pythagorean Identity to show that $\frac{dm}{d\theta} = [\sec(\theta)]^2$.

Homework.

Use the general rules and specific derivative formulas to differentiate the following functions.

(1) $f(x) = \frac{\sqrt{x}\sec(x)}{e}$

(2) $g(u) = u\cos(u) + 2\tan(u)$

(3) $h(v) = \frac{\sin(v)\cos(v)}{4}$

(4) $f(x) = 4 \cdot 3^x\cos(x)$

(5) $g(u) = \sqrt{2}(\sec(u))^2$

(6) $h(v) = \frac{1}{\sin(v)}$

(7) $f(x) = x^2e^x\tan(x)$

(8) $g(u) = \frac{1-\sin(u)}{\tan(u)}$

(9) $h(v) = \frac{1}{\tan(v)}$

Problem 10. Construct the formula for $a = f''(x)$ for the function $y = f(x) = x^3 + 5^x\cos(x)$.

Problem 11. Construct the formula for the tangent line to the graph of the function

$$y = f(x) = 5 \cdot e^x - \frac{1 + \sin(x)}{1 - \sin(x)}$$

at the point $(0, f(0))$.

Answers to the Homework.

(1) $f'(x) = \left(\frac{\sec(x)}{e}\right)\left(\frac{1}{2\sqrt{x}} + \sqrt{x}\tan(x)\right)$

(2) $g'(u) = \cos(u) - u\sin(u) + 2\sec^2(u)$

(3) $h'(v) = \frac{\cos^2(v)-\sin^2(v)}{4}$

(4) $f'(x) = 4 \cdot 3^x(\cos(x)\ln(3) - \sin(x))$

(5) $g'(u) = 2\sqrt{2}\sec(u)\tan(u)$

(6) $h(v) = -\csc(v)\cot(v)$

(7) $f'(x) = xe^x([x+2]\tan(x) + x\sec^2(x))$

(8) $g'(u) = \frac{(\sin(u)-1)\sec^2(u)-\tan(u)\cos(u)}{\tan^2(u)}$

(9) $h(v) = -\csc^2(v)$

Problem 10. Observe that

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x^3 + 5^x\cos(x)] \\ &= \frac{d}{dx}[x^3] + \frac{d}{dx}[5^x]\cos(x) + 5^x\frac{d}{dx}[\cos(x)] = 3x^2 + 5^x(\cos(x)\ln(5) - \sin(x)) \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{d}{dx}[3x^2 + 5^x(\cos(x)\ln(5) - \sin(x))] \\ &= 3\frac{d}{dx}[x^2] + \frac{d}{dx}[5^x](\cos(x)\ln(5) - \sin(x)) + 5^x\left[\ln(5)\frac{d}{dx}[\cos(x)] - \frac{d}{dx}[\sin(x)]\right] \\ &= 6x + 5^x[(\ln(5))^2\cos(x) - 2\ln(5)\sin(x) - \cos(x)] \end{aligned}$$

Problem 11. First, observe that $f(0) = 4$. Now, observe that

$$\begin{aligned} f'(x) &= 5\frac{d}{dx}[e^x] - \left(\frac{1}{[1-\sin(x)]^2}\right)\left([1-\sin(x)]\frac{d}{dx}[1+\sin(x)] - [1+\sin(x)]\frac{d}{dx}[1-\sin(x)]\right) \\ &= 5e^x - \frac{2\cos(x)}{[1-\sin(x)]^2} \end{aligned}$$

Therefore, we know $f'(0) = 3$; consequently, the formula for the tangent line is $y = T(x) = 3x + 4$.

Here is a list of the specific derivative formulas and general derivative rules we have developed so far.

Specific Derivative Formulas

1. If $y = f(x) = a$ is a constant function, then $f'(x) = 0$.
2. If r is a rational number and $y = f(x) = x^r$, then $f'(x) = rx^{r-1}$.
3. If $y = f(x) = a^x$ for any positive constant a, then $f'(x) = a^x \ln(a)$.
4. If $y = f(x) = \sin(x)$, then $f'(x) = \cos(x)$.
5. If $y = f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.
6. If $y = f(x) = \sec(x)$, then $f'(x) = \sec(\mathrm{x})\tan(x)$.
7. If $y = f(x) = \tan(x)$, then $f'(x) = [\sec(x)]^2$.

General Derivative Rules

Constant Multiple Rule
If $y = f(x)$ is a differentiable function and K is any constant, then $\frac{d}{dx}[Kf(x)] = K\frac{d}{dx}[f(x)]$.

Sum Rule
If $y = f(x)$ and $y = g(x)$ are differentiable functions, then $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$.

Product Rule
If $y = f(x)$ and $y = g(x)$ are differentiable functions, then
$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Reciprocal Rule
If $y = f(x)$ is a differentiable function, then
$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = -\frac{1}{f(x) \cdot f(x)}\frac{d}{dx}[f(x)]$$

Quotient Rule
If $y = f(x)$ and $y = g(x)$ are differentiable functions, then
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{1}{g(x) \cdot g(x)}\left[g(x) \cdot \frac{d}{dx}[f(x)] - f(x) \cdot \frac{d}{dx}[g(x)]\right]$$