Up to now, we have determined the instantaneous rate of change for functions *only* with respect to the input variable for the function. In applications, however, it is frequently necessary to determine such rates of change with respect to *other* variables. The following general derivative rule, which is one of the most important in calculus, tells us how this can be accomplished.

The Chain Rule for Derivatives

If f is a differentiable function of the input variable u and u is a differentiable function of the input variable x, then

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Basic algebra can be used to motivate why this result should be true. Let $y = f(u)$ and let $u = m(x)$ be functions and recall that the *composite* function $f \circ m$ is defined by the rule

$$[f \circ m](x) = f(m(x))$$

Observe that on the input interval from x to $x + h$, the average rate of change function for the function m is given by the formula

$$g_x(h) = \frac{m(x+h) - m(x)}{h} = \frac{\Delta u}{h}$$

(Here we have let $\Delta u = m(x+h) - m(x)$.) The average rate of change function for the function f *with respect to* x on the input interval from x to $x + h$ is given by the formula

$$G_x(h) = \frac{f(m(x+h)) - f(m(x))}{h} = \frac{f(u + \Delta u) - f(u)}{h}$$

(Here we have used the fact that if $\Delta u = m(a+h) - m(a)$, then $m(a+h) = u + \Delta u$.) Now, observe that

$$G_a(h) = \frac{f(u+\Delta u) - f(u)}{h} = \frac{f(u+\Delta u) - f(u)}{h} \cdot \left(\frac{\Delta u}{\Delta u}\right) = \left(\frac{f(u+\Delta u) - f(u)}{\Delta u}\right) \cdot \left(\frac{\Delta u}{h}\right)$$

(at least as long as $\Delta u \neq 0$). Since we have assumed that $u = m(x)$ is differentiable, we know that the function m must be *continuous*. Therefore, as the values of h approach 0, it must also be true that the values of $\Delta u = m(a+h) - m(a)$ must also approach 0. With this in mind, we know

$$\frac{df}{dx} = \lim_{h \to 0} \left(\frac{f\big(m(x+h)\big) - f\big(m(x)\big)}{h}\right)$$

$$= \lim_{h \to 0} \left[\frac{f(u+\Delta u) - f(u)}{\Delta u} \cdot \frac{\Delta u}{h}\right] = \left(\lim_{\Delta u \to 0} \frac{f(u+\Delta u) - f(u)}{\Delta u}\right) \cdot \left(\lim_{h \to 0} \frac{m(x+h) - m(x)}{h}\right) = \frac{df}{du} \cdot \frac{du}{dx}$$

The equations above provide an algebraic motivation for the Chain Rule; however, the details behind a formal justification will be left for a course on the theory of calculus.

Technical details of its proof notwithstanding, the beauty of the Chain Rule lies in the fact that we may apply it to implicitly and explicitly related quantities.

Example 1. Differentiate the function $f(x) = \tan(x)$ with respect to the variable t.

Solution. We treat the input variable x as being *implicitly* related to the variable t. In particular, we assume that x is a differentiable function of the variable t, but we just don't have enough information to determine the formula for the function. The Chain Rule tells us

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} = \frac{d}{dx}[\tan(x)] \cdot \frac{dx}{dt} = \sec^2(x)\frac{dx}{dt}$$

Since we don't have the formula that relates the values of x to the values of t, we are done.

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Problem 1. Differentiate the function $f(x) = x\sin(x)$ with respect to the variable t.

Problem 2. Consider the function $f(u) = \sqrt{u}$.

Part (a). Differentiate this function with respect to the variable x.

Part (b). Suppose we also know that $u(x) = x + \cos(x)$. Write the formula for $\frac{df}{dx}$ in terms of x.

Part (c). Now, suppose we know that $u(x) = 3 - 2^x$. Write the formula for $\frac{df}{dx}$ in terms of x.

Example 2. Use the Chain Rule to differentiate $h(x) = \sec^3(x)$ with respect to the variable x.

Solution. First, observe that $h(x) = [\sec(x)]^3$; consequently, h is constructed by composing two differentiable functions, namely

$$u(x) = \sec(x) \qquad \text{and} \qquad f(u) = u^3$$

Since $h(x) = f(u(x))$, the Chain Rule therefore tells us

$$\frac{dh}{dx} = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

The specific derivative formulas tell us

$$\frac{df}{du} = \frac{d}{du}[u^3] = 3u^2 \qquad \textit{and} \qquad \frac{du}{dx} = \frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

Thus, we know

$$\frac{dh}{dx} = (3u^2)(\sec(x)\tan(x)) = 3\sec^3(x)\tan(x)$$

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Problem 3. Consider the function $h(x) = \cos(x^{-4})$.

Part (a). Identify a function u of the variable x and a function f of the variable u so that we have $h(x) = f\big(u(x)\big)$.

Part (b). Use the Chain Rule and Part (a) to differentiate h with respect to the variable x.

$$\frac{du}{dx} = \qquad\qquad\qquad \frac{df}{du} =$$

$$\frac{dh}{dx} =$$

Problem 4. Consider the function $h(x) = [x^2 + 3\sin(x)]^4$.

Part (a). Identify a function u of the variable x and a function f of the variable u so that we have $h(x) = f\big(u(x)\big)$.

Part (b). Use the Chain Rule and Part (a) to differentiate h with respect to the variable x.

$$\frac{du}{dx} = \qquad\qquad \frac{df}{du} =$$

$$\frac{dh}{dx} =$$

Homework.

Problem 1. Differentiate the function $f(x) = \frac{2x+\sqrt{x}}{\pi}$ with respect to the variable t.

Problem 2. Differentiate the function $f(u) = u\tan(u)$ with respect to the variable x.

Each of the following functions is composite with respect to the input variable x. Use the Chain Rule to differentiate each of these functions with respect to x.

(3) $h(x) = \sqrt{\tan(x)}$

(4) $h(x) = \cos^{-4}(x)$

(5) $h(x) = \sec(3x + x^3)$

(6) $h(x) = 4^{2+\sin(x)}$

(7) $h(x) = 7(e^x + x)^{3/5}$

(8) $h(x) = \sin(\sin(x))$

Problem 9. Differentiate the function $f(x) = x^2\tan(x^3)$ with respect to the variable x.

Problem 10. Differentiate the function $f(x) = \frac{\cos(x^2)}{x}$ with respect to the variable x.

Problem 11. Let $f(x) = (x^2 - 6x + 8)^{2/3}$.

Part (a). Use the Chain Rule to construct the derivative function $r = f'(x)$.

Part (b). For what values of the input variable x will $f'(x) = 0$?

Part (c). For what values of the input variable x will the derivative function be undefined?

Answers to the Homework.

Problem 1. Differentiate the function $f(x) = \frac{2x+\sqrt{x}}{\pi}$ with respect to the variable t.

$$\frac{df}{dt} = \left(\frac{4\sqrt{x}+1}{2\sqrt{x}\pi}\right)\frac{dx}{dt}$$

Problem 2. Differentiate the function $f(u) = u\tan(u)$ with respect to the variable x.

$$\frac{df}{dx} = (\tan(u) + u\sec^2(u))\frac{du}{dx}$$

(3) $h'(x) = \frac{\sec^2(x)}{2\sqrt{\tan(x)}}$

(4) $h'(x) = 4\cos^{-5}(x)\sin(x)$

(5) $h'(x) = 3(1+x^2)\sec(3x+x^3)\tan(3x+x^3)$

(6) $h'(x) = \cos(x)4^{2+\sin(x)}\ln(4)$

(7) $h'(x) = \frac{21(1+e^x)}{5}(e^x+x)^{-2/5}$

(8) $h'(x) = \cos(x)\cos(\sin(x))$

Problem 9. Differentiate the function $f(x) = x^2\tan(x^3)$ with respect to the variable x.

$$\frac{df}{dx} = \frac{d}{dx}[x^2]\tan(x^3) + x^2\frac{d}{dx}[\tan(x^3)]$$

$$= \frac{d}{dx}[x^2]\tan(x^3) + x^2\frac{d}{du}[\tan(u)]\frac{du}{dx} \qquad \text{(Let } u = x^3\text{)}$$

$$= 2x\tan(x^3) + 3x^4\sec^2(x)$$

Problem 10. Differentiate the function $f(x) = \frac{\cos(x^2)}{x}$ with respect to the variable x.

$$\frac{df}{dx} = \left(\frac{1}{x^2}\right)\left(x\frac{d}{dx}[\cos(x^2)] - \frac{d}{dx}[x]\cos(x^2)\right)$$

$$= \left(\frac{1}{x^2}\right)\left(x\frac{d}{du}[\cos(u)]\frac{du}{dx} - \frac{d}{dx}[x]\cos(x^2)\right) \qquad \text{(Let } u = x^2\text{)}$$

$$= -\frac{2x^2\sin(x^2) + \cos(x^2)}{x^2}$$

Problem 11. Let $f(x) = (x^2 - 6x + 8)^{2/3}$.

Part (a). Use the Chain Rule to construct the derivative function $r = f'(x)$.

$$\frac{df}{dx} = \frac{d}{du}\left[u^{2/3}\right]\frac{du}{dx} \qquad \textit{Let } u = x^2 - 6x + 8$$

$$= \frac{4(x-3)}{3(x^2 - 6x + 8)^{1/3}}$$

Part (b). For what values of the input variable x will $f'(x) = 0$?

We will have $f'(x) = 0$ only when $4(x - 3) = 0$; that is, only when $x = 3$.

Part (c). For what values of the input variable x will the derivative function be undefined?

The derivative function will be undefined only when $3(x^2 - 6x + 8)^{1/3} = 0$. This occurs only when $x^2 - 6x + 8 = 0$. Now, $x^2 - 6x + 8 = (x - 2)(x - 4)$. Hence, we know $x^2 - 6x + 8 = 0$ only when $x = 2$ or $x = 4$.