The Chain Rule is one of the most useful procedures in calculus. In this investigation, we will explore two ways the Chain Rule can be used. We start by developing another important specific derivative formula.

The base-a logarithm function is defined to the inverse function for the base-a exponential function. In particular, the base-a logarithm function is defined by the relationship

$$y = g(x) = \log_a(x) \quad \Leftrightarrow \quad x = f(y) = a^y$$

Logarithm functions play an important role in many branches of science. In particular, many well-known scientific measuring scales are based on logarithm functions, including the Richter scale for measuring earthquake intensity, and the decibel scale for measuring sound intensity.

The relationship between the base-a logarithm function and the base-a exponential function enables us to use the Chain Rule to construct a formula for the derivative function for the base-a logarithm. The key is to start with the equation and differentiate with respect to the variable x. Observe

$$x = a^y \quad \Rightarrow \quad \frac{d}{dx}[x] = \frac{d}{dx}[a^y]$$

$$\Rightarrow \quad 1 = a^y \ln(a) \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{a^y \ln(a)} = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{x \ln(a)} = \frac{d}{dx}\left[\log_a(x)\right]$$

Derivative Formula for Logarithmic Functions

If a is any positive constant, then the derivative function for $y = f(x) = \log_a(x)$ is the function defined by

$$r = f'(x) = \frac{1}{x\ln(a)}$$

Problem 1. What is the formula for the line tangent to the graph of $y = f(x) = \log_2(x)$ at the point $(8, f(8))$?

Problem 2. Recall that the base-e logarithm is called the *natural* logarithm, and it is traditional to let $\ln(x)$ represent $\log_e(x)$. With this in mind, construct the formula for derivative function for the function $y = f(x) = x\ln(x) - x$.

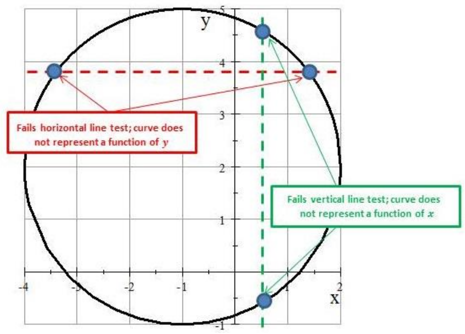
First Application --- Finding Tangent Lines to Curves

The Chain Rule can be used to determine the instantaneous rate of change in one variable with respect to another for curves, even when one of the related variables cannot be expressed as a function of the other.

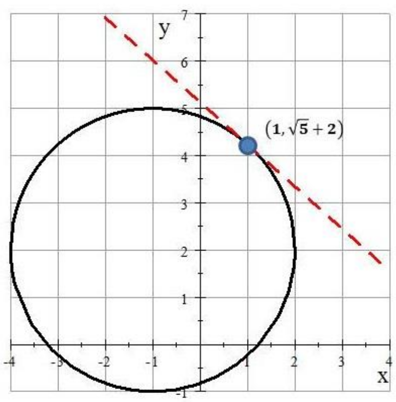
For example, the formula for a circle of radius 3 centered at the point $(-1,2)$ is

$$(x+1)^2 + (y-2)^2 = 9$$

The graph of the circle fails both the horizontal and vertical line tests. Consequently, the formula for this circle describes a relationship between x and y that cannot be made explicit with respect to either variable.



Problem 3. Use the diagram below to estimate the instantaneous rate of change in the variable y with respect to the variable x at the point $(1, \sqrt{5}+2)$ on the circle.



Let's see how we could use the Chain Rule to answer Problem 1. First, note that the instantaneous rate of change in the values of y with respect to the values of x is represented by the symbol

$$\frac{dy}{dx}$$

Consider the formula $(x+1)^2 + (y-2)^2 = 9$. If we momentarily ignore the fact that this relationship is not a function of the variable x and differentiate with respect to x, we obtain

$$(x+1)^2 + (y-2)^2 = 9 \quad \Rightarrow \quad \frac{d}{dx}[(x+1)^2 + (y-2)^2] = \frac{d}{dx}[9]$$

$$\Rightarrow \quad \frac{d}{dx}[(x+1)^2] + \frac{d}{dx}[(y-2)^2] = 0$$

Now, the Chain Rule tells us that

$$\frac{d}{dx}[(x+1)^2] = \frac{df}{du} \cdot \frac{du}{dx} \qquad \text{where } u = x+1 \text{ and } f(u) = u^2$$

$$= \frac{d}{du}[u^2] \cdot \frac{d}{dx}[x+1]$$

$$= 2u \cdot (1)$$

$$= 2(x+1)$$

We cannot write y as an explicit function of x, but we do know that the values of x and y covary in this formula. We can assume an *implicit* relationship between x and y. Now, observe

$$\frac{d}{dx}[y-2] = \frac{d}{dx}[y] - \frac{d}{dx}[2] = \frac{dy}{dx}$$

With this in mind, the Chain Rule tells us

$$\frac{d}{dx}[(y-2)^2] = \frac{df}{du} \cdot \frac{du}{dx} \qquad \text{where } u = y-2 \text{ and } f(u) = u^2$$

$$= \frac{d}{du}[u^2] \cdot \frac{d}{dx}[y-2]$$

$$= 2u \cdot \frac{dy}{dx}$$

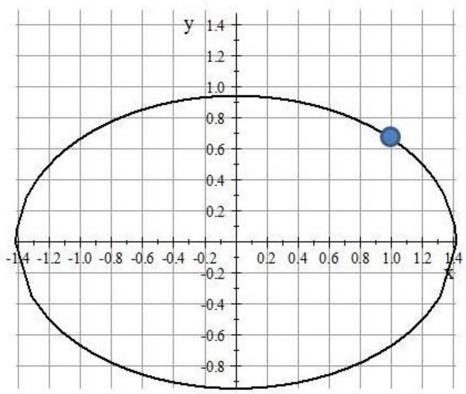
$$= 2(y-2) \cdot \frac{dy}{dx}$$

Problem 4. Consider the equation $\frac{d}{dx}[(x+1)^2] + \frac{d}{dx}[(y-2)^2] = 0$.

Part (a). Rewrite this equation using the information above and solve for the unknown $\frac{dy}{dx}$.

Part (b). At the point $(1, \sqrt{5}+2)$ on the circle, we know that $x = 1$ and $y = \sqrt{5}+2$. Use this information and your answer to Part (a) to determine the exact value of the instantaneous rate of change in the values of y with respect to the values of x.

Problem 5. Consider the ellipse defined by the formula $4x^2 + 9y^2 = 8$. The graph of this ellipse is shown below.



Part (a). Carefully sketch the graph of the tangent line to the graph of the ellipse at the point $\left(1, \frac{2}{3}\right)$. Use your graph to estimate the slope of this tangent line.

Part (b). Find a formula for $\frac{dy}{dx}$ and use this formula to find the exact value of the slope for the tangent line at this point.

Problem 6. Consider the formula $xy - y^3 = 6$.

Part (a). Construct a formula for $\frac{dy}{dx}$.

Part (b). The point (7,2) lies on the graph of the curve defined by $xy - y^3 = 6$. Construct the formula for the line tangent to the graph of the curve at this point.

Second Application --- Related Rates Problems

We frequently use the Chain Rule to find the instantaneous rate of change of one variable with respect to another when the covariance between the quantity measures represented by these variables is implicit. In the following problems, we will explore one way to use this procedure.

Problem 7. The volume V of a sphere depends on the radius R of the sphere in the following way:

$$V = f(R) = \frac{4\pi R^3}{3}$$

Suppose that the values of V and R covary implicitly with the values of a third variable t. What formula will give the *instantaneous rate of change* in the values of V with respect to the values of t?

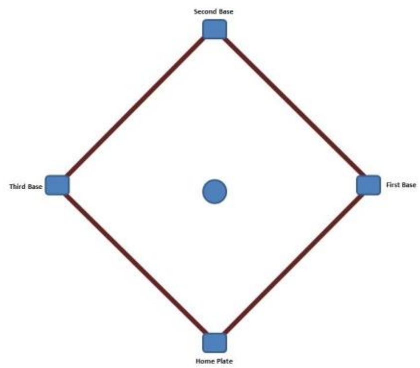
Problem 8. Imagine a square that is increasing in size. Let t represent the number of seconds elapsed since the square began increasing in size, and suppose the area of the square is measured in square inches.

Part (a). Write a formula that tells how the area of the square depends on the length of its sides. Be sure to properly define your variables.

Part (b). Take the formula you constructed in Part (a) and differentiate it with respect to the variable t. What is the practical interpretation for this formula?

Part (c). Suppose we know that, when the square is three inches wide, the instantaneous rate of change for the area of the square with respect to the values of t is 4.5 square inches per second. What is the corresponding instantaneous rate of change in the width of the square?

Problem 9. A softball diamond is a square with each side 60 feet in length. At the exact moment the ball is hit, Lauren runs from first base to second base at a constant rate of 17.6 feet per second. Let x represent Lauren's distance from first base, measured in feet, let h represent Lauren's distance from home plate, measured in feet, and let t represent the number of seconds elapsed since the ball was hit.



Part (a). Place Lauren on the baseball diamond somewhere between first and second base. Draw a line representing Lauren's distance to home plate and use your diagram to help construct a formula that gives the values of h in terms of the values of x.

Part (b). What formula gives the instantaneous rate of change for the values of h with respect to the values of t?

Part (c). What is the value of $\frac{dh}{dt}$ when Lauren is 21.6 feet from second base?

Homework.

Problem 1. Differentiate the function $h(x) = \tan(\log_5(x))$.

Problem 2. What is the second derivative of the function $y = f(x) = \ln(\cos(x))$?

Problem 3. Consider the function $y = f(x) = \log_3(x^3 - 3x + 4)$. At what values of the input variable x will the tangent line to this function be horizontal?

Problem 4. Consider the curve defined by the formula $x^2 + y^2 - 4xy = -3$.

Part (a). Construct the formula for $\frac{dy}{dx}$.

Part (b). What is the formula for the line tangent to the graph of this curve at the point (4 , 1)?

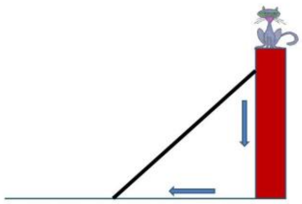
Problem 5. Consider the curve defined by the formula $y\sin(x) = 4x\cos(y)$.

Part (a). Construct the formula for $\frac{dy}{dx}$.

Part (b). What is the instantaneous rate of change in the values of y with respect to the values of x at the point $(\pi/2\ ,\ 2\pi)$?

Problem 6. Justo has a twelve-foot ladder leaning against a vertical wall as shown below. Mr. Hipsickles scampers up the ladder, causing the base of the ladder to slide away from the wall. Let x represent the horizontal distance, measured in feet, between the base of the ladder and the wall; and let y represent the vertical distance, measured in feet, between the top of the ladder and the ground. Let t represent the number of seconds passed since the ladder began to slide.

Part (a). Construct a formula that relates the values of x to the values of y.



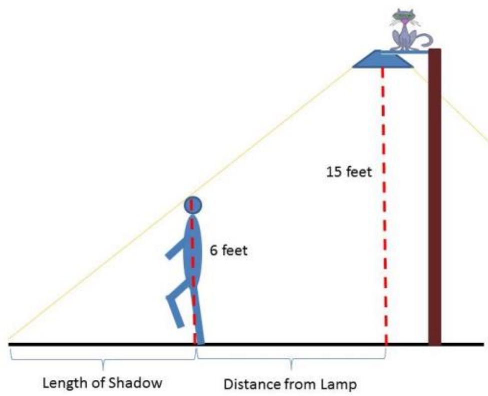
Part (b). When the base of the ladder is six feet from the wall, suppose we know that $\frac{dx}{dt} = 7.6$ feet per second. What is the corresponding value of $\frac{dy}{dt}$?

Problem 7. A cylindrical tank with a 5-foot radius is being filled with water, and the volume V of water in the tank (measured in cubic feet) is increasing at a constant rate of 2.5 cubic feet per minute. Let h represent the height of water in the tank (measured in feet) and let t represent the number of minutes since the tank began to fill.

Part (a). Go to the Internet and look up the formula that relates the volume of a cylinder to its height and radius.

Part (b). What is the value of $\frac{dh}{dt}$ when the depth of water in the tank is 3.5 feet?

Problem 8. Justo's friend, Bart Vader, is looking for Mr. Hispickles. Bart starts under a street lamp that is mounted 15 feet high on a pole that rises vertically from a stretch of level ground. Bart, who is six feet tall, walks in a straight line away from the base of the pole. When Bart is 40 feet from the pole, the horizontal distance x between Bart and the lamp (measured in feet) is increasing at 5.6 feet per second. How fast is the length of Bart's shadow (measured in feet) increasing at this moment?



Answers to the Homework.

Problem 1. Differentiate the function $h(x) = \tan(\log_5(x))$.

$$h'(x) = \frac{\sec^2(\log_5(x))}{x\ln(5)}$$

Problem 2. What is the second derivative of the function $y = f(x) = \ln(\cos(x))$?

$$h'(x) = -\tan(x) \qquad\qquad h''(x) = \sec^2(x)$$

Problem 3. Consider the function $y = f(x) = \log_3(x^3 - 3x + 4)$. At what values of the input variable x will the tangent line to this function be horizontal?

$$f'(x) = \frac{3x^2 - 3}{(x^3 - 3x + 4)\ln(3)}$$

$$f'(x) = 0 \quad \Rightarrow \quad 3x^2 - 3 = 0 \quad \Rightarrow \quad x = \pm 1$$

Problem 4. Consider the curve defined by the formula $x^2 + y^2 - 4xy = -3$.

Part (a). Construct the formula for $\frac{dy}{dx}$.

$$x^2 + y^2 - 4xy = -3 \quad \Rightarrow \quad \frac{d}{dx}[x^2 + y^2 - 4xy] = \frac{d}{dx}[3]$$

$$\Rightarrow \quad \frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] - 4\left(\frac{d}{dx}[x]y + x\frac{d}{dx}[y]\right) = \frac{d}{dx}[3]$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - 4y - 4x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2x - 4y = (4x - 2y)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{x - 2y}{2x - y} = \frac{dy}{dx}$$

Part (b). What is the formula for the line tangent to the graph of this curve at the point (4 , 1)?

At the point (4,1), we know the slope of the tangent line will be

$$\left.\frac{dy}{dx}\right|_{(x,y)=(4,1)} = \frac{4 - 2(1)}{2(4) - 1} = \frac{2}{7}$$

The formula for the tangent line will therefore be $y = \frac{2}{7}(x - 4) + 1$.

Problem 5. Consider the curve defined by the formula $y\sin(x) = 4x\cos(y)$.

Part (a). Construct the formula for $\frac{dy}{dx}$.

$$y\sin(x) = 4x\cos(y) \implies \frac{d}{dx}[y\sin(x)] = \frac{d}{dx}[4x\cos(y)]$$

$$\implies \frac{d}{dx}[y]\sin(x) + y\frac{d}{dx}[\sin(x)] = 4\left(\frac{d}{dx}[x]\cos(y) + x\frac{d}{dx}[\cos(y)]\right)$$

$$\implies \sin(x)\frac{dy}{dx} + y\cos(x) = 4\cos(y) - x\sin(y)\frac{dy}{dx}$$

$$\implies \left(\sin(x) + 4x\sin(y)\right)\frac{dy}{dx} = 4\cos(y) - y\cos(x)$$

$$\implies \frac{dy}{dx} = \frac{4\cos(y) - y\cos(x)}{\sin(x) + 4x\sin(y)}$$

Part (b). What is the instantaneous rate of change in the values of y with respect to the values of x at the point $(\pi/2\ , 2\pi)$?

$$\left.\frac{dy}{dx}\right|_{(x,y)=(\pi/2,2\pi)} = \frac{4\cos(2\pi) - (2\pi)\cos(\pi/2)}{\sin(\pi/2) + 4\left(\frac{\pi}{2}\right)\sin(2\pi)} = 4$$

Problem 6. Justo has a twelve-foot ladder leaning against a vertical wall as shown below. Mr. Hipsickles scampers up the ladder, causing the base of the ladder to slide away from the wall. Let x represent the horizontal distance, measured in feet, between the base of the ladder and the wall; and let y represent the vertical distance, measured in feet, between the top of the ladder and the ground. Let t represent the number of seconds passed since the ladder began to slide.

Part (a). Construct a formula that relates the values of x to the values of y.

$$x^2 + y^2 = 144 \quad \text{OR} \quad y = \sqrt{144 - x^2}$$

Part (b). When the base of the ladder is six feet from the wall, suppose we know that $\frac{dx}{dt} = 7.6$ feet per second. What is the corresponding value of $\frac{dy}{dt}$?

The first formula in Part (a) above is much easier to work with in this context. Observe

$$x^2 + y^2 = 144 \quad \implies \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \implies \quad \frac{dy}{dt} = -\left(\frac{x}{y}\right)\frac{dx}{dt}$$

Now, when the base of the ladder is six feet from the wall, we know $x = 6$ feet and $y = 6\sqrt{3}$ feet. Therefore,

$$\frac{dy}{dt} = -\left(\frac{1}{\sqrt{3}}\right)\left(7.6\,\frac{\text{feet}}{\text{second}}\right) \approx -4.388 \text{ feet per second}$$

Problem 7. A cylindrical tank with a 5-foot radius is being filled with water, and the volume V of water in the tank (measured in cubic feet) is increasing at a constant rate of 2.5 cubic feet per minute. Let h represent the height of water in the tank (measured in feet) and let t represent the number of minutes since the tank began to fill.

Part (a). Go to the Internet and look up the formula that relates the volume of a cylinder to its height and radius.

$$V = 25\pi h$$

Part (b). What is the value of $\frac{dh}{dt}$ when the depth of water in the tank is 3.5 feet?

$$\frac{dV}{dt} = (25\pi)\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{1}{(25\pi \text{ feet}^2)}\frac{dV}{dt}$$

$$\implies \frac{dh}{dt} = \frac{1}{(25\pi \text{ feet}^2)}\left(2.5\frac{\text{feet}^3}{\text{min}}\right) \approx 0.0318 \text{ feet per minute}$$

Notice that the rate of change for h does not depend on the actual depth of the water.

Problem 8. Justo's friend, Bart Vader, is looking for Mr. Hispickles. Bart starts under a street lamp that is mounted 15 feet high on a pole that rises vertically from a stretch of level ground. Bart, who is six feet tall, walks in a straight line away from the base of the pole. When Bart is 40 feet from the pole, the horizontal distance x between Bart and the lamp (measured in feet) is increasing at 5.6 feet per second. How fast is the length of Bart's shadow (measured in feet) increasing at this moment?

Let L represent the length of Bart's shadow, measured in feet from Bart. Let x represent the horizontal distance between from the lamp to Bart, measured in feet; and let t represent the number of seconds since Bart started walking away from the pole. Now, the relationship between the sides of similar triangles tells us

$$\frac{6}{y} = \frac{15}{x+y} \quad \implies 6x = 9y \quad \implies 6\frac{dx}{dt} = 9\frac{dy}{dt}$$

Based on the last equation, we know that, when Bart is 40 feet from the pole,

$$\frac{dy}{dt} = \frac{2}{3}\left(5.6\frac{\text{feet}}{\text{second}}\right) \approx 3.73 \text{ feet per second}$$