Up to now, we have been concerned with constructing the derivative function for a known function f. In the sciences, it is often more important to *go the other way* --- given the derivative function for some unknown function f, construct the function itself. As we shall see, this is a trickier process.

Problem 1. Show that the derivative function for $y = F(x) = x\ln(x) - x$ is the function defined by $r = F'(x) = \ln(x)$.

Problem 2. The function $y = f(x) = \ln(x)$ is only defined when $x > 0$.

Part (a). Explain why the function $y = F(x) = \ln|x|$ is defined for all nonzero values of x.

Part (b). Explain why, when $x < 0$, we have $\ln|x| = \ln(-x)$.

Part (c). Use the Chain Rule to determine the formula for $\frac{d}{dx}[\ln(-x)]$.

Problem 3. Explain why $\frac{d}{dx}[\ln|x|] = \frac{1}{x}$ for all nonzero values of x.

Antiderivative

Let $r = f(x)$ be a function. We say that a function $y = F(x)$ is an *antiderivative* for the function f provided

$$F'(x) = f(x)$$

In Problems 1 - 3, you showed that

$$y = F(x) = x\ln(x) - x \text{ is one antiderivative for the function } r = f(x) = \ln(x)$$

$$y = F(x) = \ln|x| \text{ is one antiderivative for the function } r = f(x) = \frac{1}{x}$$

Problem 4. Suppose you know that $y = F(x)$ is one antiderivative for a function $r = f(x)$. If C represents any constant, explain why the function

$$y = F_C(x) = F(x) + C$$

is also an antiderivative for the function f. (What happens if you differentiate the function F_C?)

The Antiderivative Family for a Function

Let $r = f(x)$ be a function. If $y = F(x)$ and $y = G(x)$ are antiderivatives for the function f, then

$$G(x) - F(x) = C$$

for some constant C. It is customary to use the symbol

$$\int f(x)dx$$

to represent the family of *all* antiderivatives for the function f.

This fact is yet another consequence of the Mean Value Theorem. To see why it is valid, let $y = h(x) = F(x) - G(x)$. Since both F and G are differentiable (we have assumed that f is the derivative function for both), we see that

$$h'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$$

We want to show that the function h has constant output. This means we want to show that for *any* distinct input values $x = a$ and $x = b$, we have $h(a) = h(b)$. To this end, let $x = a$ and $x = b$ be distinct input values. The Mean Value Theorem tells us there is some input value $a < c < b$ such that

$$0 = h'(c) = \frac{h(b) - h(a)}{b - a}$$

It follows that $0 = h(b) - h(a)$; and we must conclude that $h(a) = h(b)$, as desired. It now follows that there is some constant C such that

$$C = h(x) = F(x) - G(x)$$

Based on Problem 1, we know that $F(x) = x\ln(x) - x$ is *one* of the antiderivatives for the function $f(x) = \ln(x)$. The Mean Value Theorem tells us that we can construct *any* antiderivative for the function f simply by adding a constant to the function F.

$$\int \ln(x)\,dx = \{G(x) = x\ln(x) - x + C : C \text{ is any constant}\}$$

This notation is read "*The antiderivative family for the function $f(x) = \ln(x)$ is the set of all functions $G(x) = x\ln(x) - x + C$ such that C is any constant.*"

It is customary (but not really correct) to abbreviate the notation above to

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

The abbreviated notation is still read the same way.

Problem 5. Use the customary notation to write the antiderivative families for the two functions $f(x) = \sec^2(x)$ and $(x) = \frac{1}{x}$.

$$\int \sec^2(x)\,dx = \qquad\qquad\qquad \int \frac{1}{x}\,dx =$$

Sometimes, we can apply the general derivative rules *in reverse* to determine antiderivative families. For example, suppose we want to know the antiderivative family for the function $r = f(x) = x^2$.

$$\frac{d}{dx}[x^3] = 3x^2 \quad \Rightarrow \quad \left(\frac{1}{3}\right)\frac{d}{dx}[x^3] = x^2$$

$$\Rightarrow \quad \frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2$$ ***(Apply the Constant Multiple Rule in reverse.)***

$$\Rightarrow \quad \int x^2\,dx = \frac{1}{3}x^3 + C$$

Problem 6. Use the fact that $\frac{d}{dx}[\cos(x)] = (-1)\sin(x)$ to find the antiderivative family for the function $r = f(x) = \sin(x)$.

Problem 7. Consider the function $y = g(x) = Kx$, where K is any constant.

Part (a). Construct the formula for $r = g'(x)$.

Part (b). Use your answer to Part (a) to determine the antiderivative family for the function $f(x) = K$ when K is any constant.

Problem 8. Consider the function $y = g(x) = x^{n+1}$, where n is any rational number other than -1.

Part (a). Use the Power Rule to construct the formula for $r = g'(x)$.

Part (b). Use your answer to Part (a) to determine the antiderivative family for the function $f(x) = x^n$ when n is any rational number other than -1. Why is it necessary to exclude $n = -1$?

At this stage, you probably are asking yourself why we don't just content ourselves with finding *one* antiderivative for a function rather than insisting that an arbitrary constant C be included as part of the antidifferentiation process. We do this because there are infinitely many different functions that serve as an antiderivative for any given function; and sometimes we need to specify the *particular* antiderivative for a function that satisfies a given condition.

Let's consider an example.

Problem 9. Samuel places an ice cube on the kitchen table, and it starts to melt. Suppose we know that the weight $W = H(t)$ of an ice cube, measured in ounces, is changing with respect to the number t of minutes passed since the ice cube was placed on the table according to the formula

$$H'(t) = 1 - \frac{10}{e^t}$$

Part (a). Consider the function $F(t) = t + 10e^{-t}$. By differentiating this function, show that F is *one* antiderivative for the function H.

Part (b). After three minutes, Samuel weighs the ice cube and finds that it weighs 7.75 ounces. Explain why this tells us that $F(t) \neq H(t)$.

Part (c). Since the function F is an antiderivative for the weight function, there must exist a constant C such that $H(t) = F(t) + C$. What is the approximate value of this constant?

Part (d). Suppose that Samuel misread the scale when he weighed the ice cube. Six minutes after placing the ice cube on the table, he re-measures the weight and determines that the ice cube weighs 5.25 ounces. Now what is the value of the constant C needed to make $H(t) = F(t) + C$?

There is something important about the relationship between the function F and the function H' that appear in Problem 9. Notice that H' is actually the *sum* of two functions, namely the function $f(t) = 1$ and the function $g(t) = -10e^{-t}$. Since

$$\frac{d}{dt}[t] = 1 \qquad \text{and} \qquad \frac{d}{dt}[10e^{-t}] = -10e^{-t}$$

we know that the antiderivative family for the functions f and g will be

$$\int 1dt = t + C_1 \qquad \text{and} \qquad \int(-10e^{-t})dt = 10e^{-t} + C_2$$

Furthermore, we also know

$$\frac{d}{dt}[t + 10e^{-t}] = 1 - \frac{10}{e^t}$$

It follows that the antiderivative family for the function H' will be

$$\int \left(1 - \frac{10}{e^t}\right) dt = t + 10e^{-t} + C$$

It therefore stands to reason that we have

$$\int \left(1 - \frac{10}{e^t}\right) dt = \int 1dt + \int (-10e^{-t})dt$$

In other words, it stands to reason that *the antiderivative family for this sum of functions is the sum of the antiderivative families of the functions.*

General Antiderivative Rules

Suppose that $r = f(x)$ and $r = g(x)$ are functions, and suppose that K is a fixed constant.

- **Anti-Constant-Multiple Rule:** $\int Kf(x)\,dx = K\int f(x)\,dx$
- **Anti-Sum Rule:** $\int [f(x) + g(x)]dx = \int f(x)\,dx + \int g(x)\,dx$

Problem 10. Use the general antiderivative rules to evaluate

$$\int \left[3\sin(x) + \frac{2}{x} - 5\right] dx$$

Special Antiderivative Formulas

1. Since $\frac{d}{dx}[Kx] = K$ for any constant K, we know $\int K\,dx = Kx + C$.
2. Since $\frac{d}{dx}[\sin(x)] = \cos(x)$, we know $\int \cos(x)\,dx = \sin(x) + C$.
3. Since $\frac{d}{dx}[\cos(x)] = -\sin(x)$, we know $\int \sin(x)\,dx = -\cos(x) + C$.
4. Since $\frac{d}{dx}[\tan(x)] = \sec^2(x)$, we know $\int \sec^2(x)\,dx = \tan(x) + C$.
5. Since $\frac{d}{dx}[x\ln(x) - x] = \ln(x)$, we know $\int \ln(x)\,dx = x\ln(x) - x + C$.
6. Since $\frac{d}{dx}[\ln|x|] = x^{-1}$, we know $\int x^{-1}\,dx = \ln|x| + C$.
7. Since $\frac{d}{dx}[x^{n+1}] = (n+1)x^n$, when $n \neq -1$, we know $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$.
8. Since $\frac{d}{dx}[a^x] = a^x\ln(a)$, we know $\int a^x\,dx = \frac{1}{\ln(a)} \cdot a^x + C$.

Problem 11. Determine the antiderivative family for the function $f(x) = 3e^x - 4\ln(x)$.

Problem 12. Evaluate $\int \left[2b^{-2} + 5b^{1/2} - 3\right] db$.

Problem 13. Find the antiderivative family for the function $m = f(n) = 3n^{-3/4} + 2\sec^2(n)$.

Homework.

Problem 1. Consider the function $f(x) = \sin(x)\cos(x)$, along with the functions

$$F(x) = \frac{\sin^2(x)}{2} \qquad G(x) = -\frac{\cos^2(x)}{2}$$

Part (a). By differentiating the functions F and G, show that *both* serve as antiderivatives for the function f.

Part (b). Since both of the functions F and G serve as antiderivatives for the function f, there must exist a constant C such that $F(x) = G(x) + C$. What is the value of this constant?

Problem 2. By differentiating the function $F(x) = \frac{e^x}{2}[\sin(x) - \cos(x)]$, show that F is one antiderivative for the function $f(x) = e^x \sin(x)$.

Problem 3. By differentiating the function $F(x) = \ln(1 + x^2) + \ln(x)$, show that F is one antiderivative for the function

$$r = f(x) = \frac{3x^2 + 1}{x + x^3}$$

Evaluate each of the following.

(4) $\int [2e^x - 3\cos(x)]dx$

(5) $\int [3\ln(t) + 5t^{-2}]\, dt$

(6) $\int \left[2v^{1/3} - \sin(v)\right] dv$

(7) $\int \left[\frac{4}{x} + \frac{3}{\sqrt{x}} + \frac{5}{2}\right] dx$

(8) $\int (3 - 2t)\, dt$

(9) $\int \left(\sec^2(v) + \frac{\pi}{v^3}\right) dv$

Problem 10. A toy car is moving back and forth along a straight track. Let s represent the distance of the car (measured in feet) from its starting point and let t represent the number of seconds passed since the car started moving. Suppose the velocity v (measured in feet per second) for the car is given by the function

$$v = h(t) = 3 + 2\cos(t)$$

Suppose that $s = 4.3$ feet when $t = 5$ seconds. What is the formula for the function $s = H(t)$ that gives the values of s in terms of the values of t?

Answers.

(1) Observe that the Chain Rule tells us

$$\frac{dF}{dx} = \frac{1}{2} \cdot \frac{d}{du}[u^2] \frac{d}{dx}[\sin(x)] = \frac{2u \cdot \cos(x)}{2} = \cos(x)\sin(x)$$

$$\frac{dG}{dx} = -\frac{1}{2} \cdot \frac{d}{du}[u^2] \frac{d}{dx}[\cos(x)] = \frac{2u \cdot \sin(x)}{2} = \cos(x)\sin(x)$$

There are many ways to approach Part (b). One way is to take advantage of the Pythagorean Identity. Observe that

$$F(x) - G(x) = \frac{\sin^2(x)}{2} + \frac{\cos^2(x)}{2} = \frac{\left(\sin^2(x) + \cos^2(x)\right)}{2} = \frac{1}{2}$$

Consequently, $F(x) = G(x) + \frac{1}{2}$.

(2) Observe that the Product Rule followed by the Sum and Constant Multiple Rules tell us

$$\frac{d}{dx}\left[\frac{e^x}{2}[\sin(x) - \cos(x)]\right] = \frac{1}{2}\left(\frac{d}{dx}[e^x] \cdot (\sin(x) - \cos(x)) + e^x\left(\frac{d}{dx}[\sin(x)] - \frac{d}{dx}[\cos(x)]\right)\right)$$

$$= \frac{e^x}{2}([\sin(x) - \cos(x)] + [\cos(x) + \sin(x)])$$

$$= e^x \sin(x)$$

(3) Observe that the Sum Rule, followed by the Chain Rule tells us

$$\frac{d}{dx}[\ln(1 + x^2) + \ln(x)] = \frac{d}{dx}[\ln(1 + x^2)] + \frac{d}{dx}[\ln(x)]$$

$$= \frac{2x}{1 + x^2} + \frac{1}{x}$$

$$= \frac{3x^2 + 1}{x + x^3}$$

(4) $\int[2e^x - 3\cos(x)]dx = 2e^x - 3\sin(x) + C$

(5) $\int[3\ln(t) + 5t^{-2}]\,dt = 3(t\ln(t) - t) - 5t^{-1} + C$

(6) $\int\left[2v^{1/3} - \sin(v)\right]dv = \frac{3}{2}v^{4/3} + \cos(v) + C$

(7) $\int\left[\frac{4}{x} + \frac{3}{\sqrt{x}} + \frac{5}{2}\right]dx = 4\ln|x| + 6\sqrt{x} + \frac{5}{2}x + C$

(8) $\int(3 - 2t)\,dt = 3t - t^2 + C$

(9) $\int\left(\sec^2(v) + \frac{\pi}{v^3}\right)dv = \tan(v) - \frac{\pi}{2v^2} + C$

(10) We know that the velocity function is the derivative of the distance function; hence, we know that the function H is one antiderivative for the function h. The antiderivative family for the function h is the set

$$\int(3 + 2\cos(t))dt = 3t + 2\sin(t) + C$$

We are told that $H(5) = 4.3$, so the formula we want must satisfy the equation

$$4.3 = 3(5) + 2\sin(5) + C$$

Solving this equation for C tells us that $H(t) \approx 3t + 2\sin(t) - 8.782$.