In the last investigation, we explored the notion of antidifferentiation. As part of this exploration, we encountered general antiderivative rules that "undo" the Sum Rule and Constant Multiple Rule for derivatives. There is also a general antiderivative rule that "undoes" the Chain Rule for derivatives, and we will explore that rule in this investigation.

The Anti-Chain Rule

Let $r = f(x)$ be a function. If it is possible to find functions $r = g(u)$ and $u = h(x)$ so that

$$f(x) = g(u) \cdot \frac{du}{dx}$$

then the following equation is true:

$$\int f(x)\,dx = \int \left[g(u) \cdot \frac{du}{dx}\right] dx = \int g(u)du$$

Example 1. Use the Anti-chain Rule to find the antiderivative family for $r = f(x) = 2x\sin(x^2)$.

Solution. If we let $u = h(x) = x^2$, then $\frac{du}{dx} = 2x$. Consequently, we know

$$\int 2x\sin(x^2)\,dx = \int \sin(u) \cdot [\mathbf{2x}]\,dx = \int \sin(u) \cdot \left[\frac{\mathbf{du}}{\mathbf{dx}}\right] dx = \int \sin(u)\,du$$

We have now recast the antiderivative problem so that it exactly matches one of the special antiderivative formulas. Therefore, we know

$$\int 2x\sin(x^2)\,dx = \int \sin(u)\,du = -\cos(u)|_{u=x^2} + C = -\cos(x^2) + C$$

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Problem 1. Use the Anti-chain Rule to find the antiderivative family for $f(x) = 4x^3\cos(x^4)$.

Problem 2. Use the Anti-chain Rule to evaluate $\int \frac{1}{x}(2 + \ln(x))^3\,dx$.

Example 2. Use the Anti-chain Rule to find the antiderivative family for $r = f(x) = x^2\sqrt{1+4x^3}$.

Solution. In this case, let $u = h(x) = 1 + 4x^3$. This tells us

$$\frac{du}{dx} = 12x^2 \qquad \text{so that} \qquad \frac{1}{12}\cdot\frac{du}{dx} = x^2$$

Consequently, we know

$$\int x^2\sqrt{1+4x^3}\,dx = \int \sqrt{u}\cdot[x^2]dx$$

$$= \int \sqrt{u}\cdot\left[\frac{1}{12}\cdot\frac{du}{dx}\right]dx$$

$$= \frac{1}{12}\int u^{1/2}\cdot\left[\frac{du}{dx}\right]dx$$

$$= \frac{1}{12}\int u^{1/2}\,du$$

$$= \left(\frac{1}{12}\right)\left(\frac{1}{1/2+1}\right)u^{1/2+1}\Big|_{u=1+4x^3} + C$$

$$= \frac{1}{8}(1+4x^3)^{3/2} + C$$

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Problem 3. Evaluate $\int x\sin(x^2)dx$.

Problem 4. Find the antiderivative family for the function $f(x) = \frac{x^2}{2+x^3}$.

Hint: Note that $f(x) = x^2(2+x^3)^{-1}$.

Homework.

Evaluate each of the following.

(1) $\int 2x\ln(x^2)dx$

(2) $\int [3t^2\cos(t^3)]\,dt$

(3) $\int \frac{v}{v^2+3}dv$

(4) $\int \left[x - x^3\sqrt{x^4-2}\right]dx$

(5) $\int t^3\ln(t^4)dt$

(6) $\int \sin(v)\sec^2(\cos(v))dv$

(7) $\int (x-1)(x^2-2x)^3dx$

(8) $\int \frac{1}{\sqrt{1-2t}}dt$

(9) $\int \frac{2v^2+3}{(2v^3+9v)^4}dv$

(10) $\int \frac{\sin(x)}{\cos(x)}dx$

(11) $\int \left[t^{-3/4} + t\sin(t^2)\right]dt$

Answers.

(1) $\int 2x\ln(x^2)dx = x^2\ln(x^2) - x^2 + C$

(2) $\int [3t^2\cos(t^3)]\,dt = \sin(t^3) + C$

(3) $\int \frac{v}{v^2+3}dv = \frac{1}{2}\ln|v^2+3| + C$

(4) $\int \left[x - x^3\sqrt{x^4-2}\right]dx = \frac{x^2}{2} - \frac{1}{6}(x^4-2)^{3/2} + C$

(5) $\int t^3\ln(t^4)dt = \frac{1}{4}[t^4\ln(t^4) - t^4] + C$

(6) $\int \sin(v)\sec^2(\cos(v))dv = -\tan(\cos(v)) + C$

(7) $\int (x-1)(x^2-2x)^3dx = \frac{1}{8}(x^2-2x)^4 + C$

(8) $\int \frac{t}{\sqrt{1-t^2}}dt = -\frac{1}{2}\sqrt{1-t^2} + C$

(9) $\int \frac{2v^2+3}{2v^3+9v}dv = \frac{1}{3}\ln|2v^3+9v| + C$

(10) $\int \frac{\sin(x)}{\cos(x)}dx = -\ln|\cos(x)| + C$

(11) $\int \left[t^{-3/4} + t\sin(t^2)\right]dt = 4t^{1/4} - \frac{1}{2}\cos(t^2) + C$