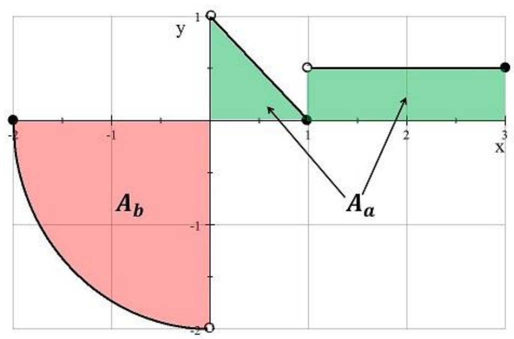
In this investigation and the ones that follow it, we will introduce an idea that seems almost trivial at first, but turns out to represent one of the great breakthroughs of human thinking. We begin with a definition.

> ***Integrable Function***
>
> Let $y = f(x)$ be a function. We say that the function f is *integrable* on the input interval $L \leq x \leq U$ provided the following conditions are met.
> - The function f has no vertical asymptotes in this input interval.
> - The function f has only a finite number of jump discontinuities and removable discontinuities in this input interval.

Suppose that $y = f(x)$ is an integrable function on some input interval $L \leq x \leq U$. We will let A_a represent the area enclosed between the graph of f and the x-axis *when the output of f is positive.* We will let A_b represent the area enclosed between the graph of f and the x-axis *when the output of f is negative.*



The diagram above shows a function f that is integrable on the input interval $-2 \leq x \leq 3$. The diagram also shows the shaded regions that correspond to the values A_a and A_b for the function on this input interval.

The function f whose graph is shown above, is piecewise defined. If we assume that the portion of the graph on the input interval $-2 \leq x \leq 0$ is one-quarter of a circle of radius 2, then

$$A_b = \left(\frac{1}{4}\right)(4\pi) = \pi$$

$$A_a = \left[\left(\frac{1}{2}\right)(1 \times 1)\right] + \left[\frac{1}{2} \times 2\right] = \frac{3}{2}$$

Area of Triangle + Area of Rectangle

> ***Net Area***
>
> Let $y = f(x)$ be a function that is integrable on the input interval $L \leq x \leq U$. The *net area* between the graph of f and the x-axis on this input interval is defined to be
>
> $$\int_L^U f(x)dx = A_a - A_b$$

For example, if we consider the function f defined by the diagram above, we know

$$\int_{-2}^{3} f(x)dx = \frac{3}{2} - \pi \approx -1.642$$

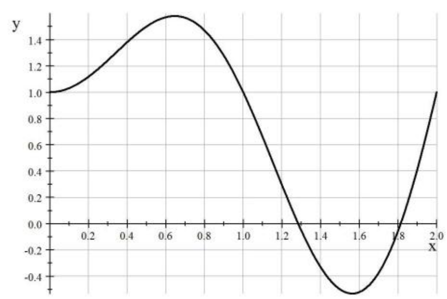
The fact that the net area for the function f on this input interval is negative simply means there is more area below the x-axis than there is area above the x-axis.

Problem 1. Let f be the function defined in the diagram above. Compute the following net areas.

$$\int_{0}^{3} f(x)dx \qquad \int_{2}^{3} f(x)dx \qquad \int_{-2}^{1} f(x)dx$$

Of course, the graphs of most integrable functions do not consist of straight line segments and arcs of circles. For most integrable functions, we can only approximate the net area between their graph and the x-axis. There are many ways we could accomplish this approximation, but one systematic approach yields some very unexpected fruit. Let's consider an example.

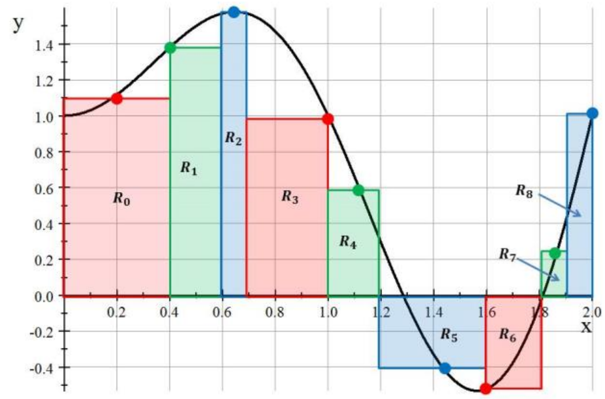
Consider the function f whose graph is shown below.



Here is one way we could go about approximating the net area between the graph of f and the x-axis on the input interval $0 \leq x \leq 2$.

1. Break up the input interval into a sequence of subintervals.
2. Select an input value at random from each subinterval and evaluate the function f at the input value you choose.
3. On each subinterval, construct a rectangle whose base is the subinterval and whose height is the output of f at the input value you chose from that subinterval.
4. Compute the "signed" area of each rectangle. (Area will be negative if the rectangle is below the x-axis.)
5. Add up the "signed" areas of the rectangles.

Example 1. Here is one way we could construct such rectangles for the function f on the input interval $0 \leq x \leq 2$.



We can use the graph to approximate the "signed" areas for each rectangle.

"Signed" area of R_0 is $A_0 \approx (0.4) \times (1.1) = 0.44$
BASE × "HEIGHT"

"Signed" area of R_1 is $A_1 \approx (0.2) \times (1.39) = 0.278$
BASE × "HEIGHT"

"Signed" area of R_2 is $A_2 \approx (0.1) \times (1.6) = 0.16$
BASE × "HEIGHT"

"Signed" area of R_3 is $A_3 \approx (0.3) \times (0.99) = 0.297$
BASE × "HEIGHT"

"Signed" area of R_4 is $A_4 \approx (0.2) \times (0.59) = 0.118$
BASE × "HEIGHT"

"Signed" area of R_5 is $A_5 \approx (0.4) \times (-0.4) = -0.16$
BASE × "HEIGHT"

"Signed" area of R_6 is $A_6 \approx (0.2) \times (-0.55) = -0.11$
BASE × "HEIGHT"

"Signed" area of R_7 is $A_7 \approx (0.1) \times (0.25) = 0.025$
BASE × "HEIGHT"

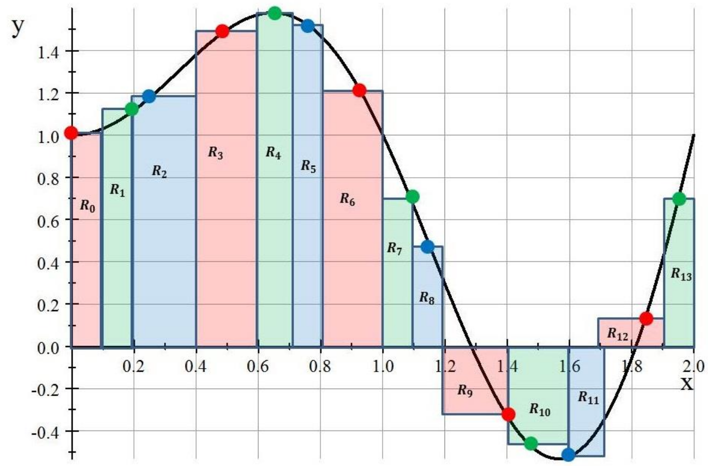
"Signed" area of R_8 is $A_8 \approx (0.1) \times (1.0) = 0.10$
BASE × "HEIGHT"

Consequently, based on our choice of rectangles, we see that

$$\int_0^2 f(x)dx \approx 0.44 + 0.278 + 0.16 + 0.297 + 0.118 - 0.16 - 0.11 + 0.025 + 0.10 \approx 1.148$$

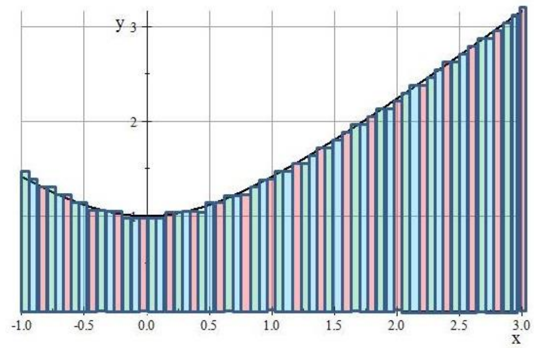
Of course, just looking at the rectangles drawn above, we can guess that this is not a very good estimate. We could do better by choosing more (and *skinnier* rectangles) that fit under the curve better.

Problem 2. Use the rectangles shown to approximate the value of $\int_0^2 f(x)dx$.



It stands to reason that, if we divide the interval $0 \leq x \leq 2$ into smaller and smaller subintervals and use these subintervals to correspondingly skinnier and skinnier rectangles, the sums of the "signed" areas of these rectangles should provide increasingly better approximations to the net area.

As a different example, here is a diagram that shows sixty approximating rectangles drawn under the curve $y = f(x) = \sqrt{1 + x^2}$ on the input interval $-1 \leq x \leq 3$.



Notice how these rectangles "fill in" the region between the curve and the x-axis. The sum of the areas of these rectangles should provide a very accurate approximation for the value of

$$\int_{-1}^{3} \sqrt{1 + x^2}\, dx$$

This suggests that there is some sort of *limiting process* involved in computing net area. To understand this process, we need to introduce some notation that allows us to describe how we go about constructing our approximating rectangles. We begin with a notational way to describe dividing the input interval into subintervals.

Partition of an Interval

Consider an interval $a \leq x \leq b$. A *partition* of this interval is a sequence of $n+1$ numbers

$$x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b$$

with the property that $x_j < x_{j+1}$ for every $0 \leq j < n$.

For example, here are two different partitions of the interval $-1 \leq x \leq 3$ that contain nine numbers.

- $x_0 = -1 \quad x_1 = -0.9 \quad x_2 = -0.75 \quad x_3 = -0.5 \quad x_4 = 0.95 \quad x_5 = 1.0 \quad x_6 = 2.9 \quad x_7 = 2.99 \quad x_8 = 3$
- $x_0 = -1 \quad x_1 = -0.5 \quad x_2 = 0.0 \quad x_3 = 0.5 \quad x_4 = 1.0 \quad x_5 = 1.5 \quad x_6 = 2.0 \quad x_7 = 2.5 \quad x_8 = 3$

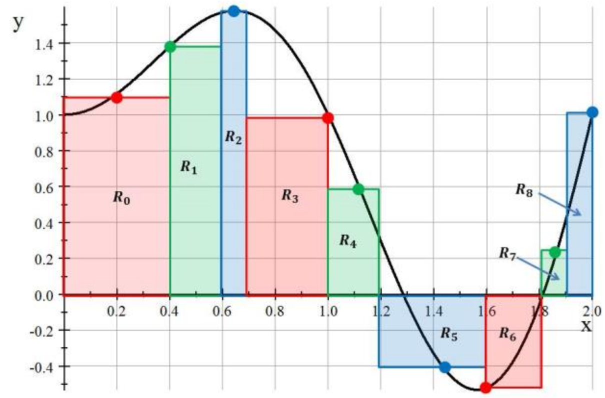
Problem 3. Construct two different partitions of the interval $0 \leq x \leq 4$ that contain eight numbers.

We can use partitions to help us describe the construction of approximating rectangles for a function f that is integrable on an input interval $a \leq x \leq b$.

1. Construct any partition $x_0 = a,\, x_1,\, x_2,\, \dots,\, x_n = b$.
2. Every pair of consecutive numbers x_j and x_{j+1} in this partition can be used as the base of a rectangle. The width of this rectangle will be $\Delta x_j = x_{j+1} - x_j$.
3. Choose any number you like from each subinterval $x_j \leq x \leq x_{j+1}$ and call this number x_j^*.
4. Evaluate the function f at the input value x_j^*. (These input values are called *tags*.)
5. Let A_j represent the "signed" area of the rectangle whose width is Δx_j and whose "signed" height is $f\left(x_j^*\right)$. In symbols, we have $A_j = f\left(x_j^*\right)\Delta x_j$.
6. The sum of these "signed" areas is an approximation to the net area. In symbols, we have

$$\int_a^b f(x)dx \approx f(x_0^*)\Delta x_0 + f(x_1^*)\Delta x_1 + \cdots + f(x_{n-1}^*)\Delta x_{n-1}$$

Problem 4. Consider the rectangles used to approximate the net area for the function $y = f(x)$ that are shown in Example 1 above.



Part (a). Identify the ten numbers that form the partition used.

$x_0 =$ ________ $x_1 =$ ________ $x_2 =$ ________ $x_3 =$ ________ $x_4 =$ ________

$x_5 =$ ________ $x_6 =$ ________ $x_7 =$ ________ $x_8 =$ ________ $x_9 =$ ________

Part (b). Each pair of consecutive numbers you listed above forms the endpoints of a subinterval used as the base of a rectangle. A tag was selected from each of these subintervals, and the corresponding output from f was used as the "signed" height of the rectangle. Identify the tag selected from each subinterval.

$x_0^* =$ ________ $x_1^* =$ ________ $x_2^* =$ ________ $x_3^* =$ ________ $x_4^* =$ ________

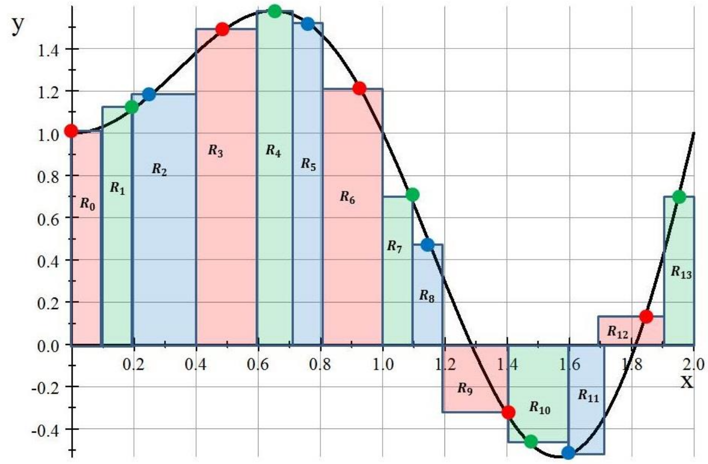
$x_5^* =$ ________ $x_6^* =$ ________ $x_7^* =$ ________ $x_8^* =$ ________

Part (c). The equation below shows the approximation obtained using the "signed" areas of the eight rectangles. Write this approximation as a sum of terms $f(x_j^*)\Delta x_j$ using your information in Parts (a) and (b).

$$\int_0^2 f(x)dx \approx \underbrace{0.44}_{A_0} + \underbrace{0.278}_{A_1} + \underbrace{0.16}_{A_2} + \underbrace{0.297}_{A_3} + \underbrace{0.118}_{A_4} + \underbrace{(-0.16)}_{A_5} + \underbrace{(-0.11)}_{A_6} + \underbrace{0.025}_{A_7} + \underbrace{0.10}_{A_8} \approx 1.148$$

(For example, we would have $A_3 = f(1.0) \cdot \Delta x_3$ since $x_3^* = 1.0$.)

Problem 5. Consider the rectangles used to approximate the net area for the function $y = f(x)$ that are shown in the diagram below.



Part (a). Identify the sixteen numbers that form the partition used.

$x_0 =$ ________ $x_1 =$ ________ $x_2 =$ ________ $x_3 =$ ________ $x_4 =$ ________

$x_5 =$ ________ $x_6 =$ ________ $x_7 =$ ________ $x_8 =$ ________ $x_9 =$ ________

$x_{10} =$ ________ $x_{11} =$ ________ $x_{12} =$ ________ $x_{13} =$ ________

$x_{14} =$ ________ $x_{15} =$ ________

Part (b). Each pair of consecutive numbers you listed above forms the endpoints of a subinterval used as the base of a rectangle. A tag was selected from each of these subintervals, and the corresponding output from f was used as the "signed" height of the rectangle. Identify the tag selected from each subinterval.

$x_0^* =$ ________ $x_1^* =$ ________ $x_2^* =$ ________ $x_3^* =$ ________ $x_4^* =$ ________

$x_5^* =$ ________ $x_6^* =$ ________ $x_7^* =$ ________ $x_8^* =$ ________ $x_9^* =$ ________

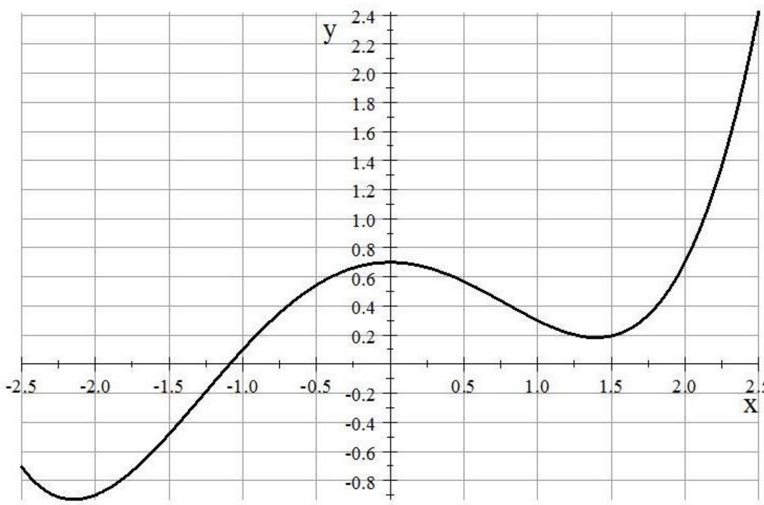
$x_{10}^* =$ ________ $x_{11}^* =$ ________ $x_{12}^* =$ ________ $x_{13}^* =$ ________

$x_{14}^* =$ ________

Part (c). Write the net area approximation as a sum of terms $f(x_j^*)\Delta x_j$ using your information in Parts (a) and (b).

$$\int_0^2 f(x)dx \approx$$

Problem 6. Consider the graph of the function $y = f(x)$ shown below.



Part (a). Using the diagrams in Problems 4 and 5 as guides, draw twelve approximating rectangles R_0, … , R_{11} whose "signed" areas could be used to approximate the net area

$$\int_{-2.5}^{2.5} f(x)dx$$

Take care to create your rectangles to obtain as good an approximation as you can. (In other words, try to minimize the overestimates and underestimates you get from your rectangles.) There are many valid approaches.

Part (b). Write down the sequence of thirteen numbers in the partition that your rectangles create. Use proper indexing.

Part (c). Write down the sequence of twelve tags created by the "signed" heights of your rectangles. Use proper indexing.

Part (d). Write the resulting approximation to the net area as a sum of terms of the form $f(x_j^*)\Delta x_j$, and then compute the approximation.

$$\int_{-2.5}^{2.5} f(x)dx \approx$$

Riemann Sum

Suppose that $y = f(x)$ is an integrable function on an input interval $a \le x \le b$. Let $x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b$ be any partition of the input interval, and for each index $0 \le j < n$, let x_j^* be any tag chosen from the input subinterval $x_j \le x \le x_{j+1}$. A *Riemann Sum* has the form

$$\sum_{j=0}^{n-1} f(x_j^*)\Delta x_j = f(x_0^*)\Delta x_0 + f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_{n-2}^*)\Delta x_{n-2} + f(x_{n-1}^*)\Delta x_{n-1}$$

where $\Delta x_j = x_{j+1} - x_j$.

As the previous problems show, we use Riemann sums to approximate net areas. While we can choose our partition numbers and tags any way that we like, it is quite common to instill some consistency on the process. For example,

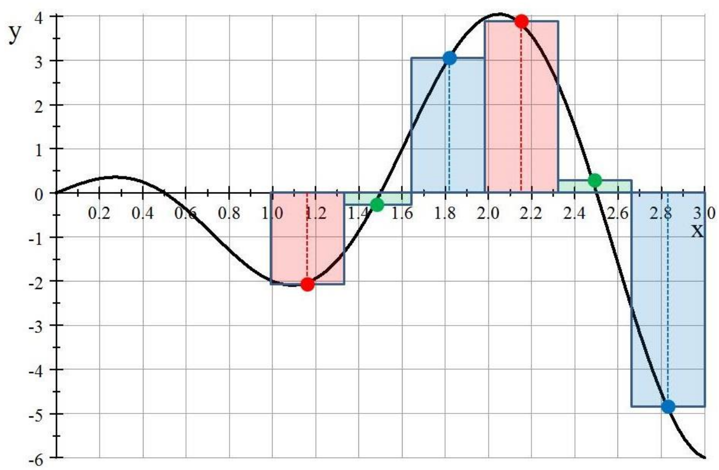
- It is common to choose our partition numbers so that they are equally spaced.
- It is common to choose our tags the same way for each subinterval. (For example, we could always let our tag be the midpoint of each subinterval.)

We create a *midpoint approximation* for a net area when we follow the two suggestions above.

Example 2. Use a midpoint approximation with six approximating rectangles to estimate the value of

$$\int_1^3 (2x\cos(\pi x))dx$$

Solution. The diagram below shows the rectangles used in this particular midpoint approximation.



First, note that each rectangle has equal width. We determine this width by dividing the width of the input interval by the number of rectangles we want to use. In this case, $w = \frac{3-1}{6} = \frac{1}{3}$.

Since each rectangle has the same width, there is a systematic way to construct the numbers in the partition.

$$x_0 = 1 \qquad x_1 = x_0 + \frac{1}{3} = \frac{4}{3} \qquad x_2 = x_1 + \frac{1}{3} = \frac{5}{3} \qquad x_3 = x_2 + \frac{1}{3} = 2$$

$$x_4 = x_3 + \frac{1}{3} = \frac{7}{3} \qquad x_5 = x_4 + \frac{1}{3} = \frac{8}{3} \qquad x_6 = x_5 + \frac{1}{3} = 3$$

We have opted to use the midpoint of each subinterval as the tag. There is a systematic way we can determine these tags as well --- we can take the left endpoint of each subinterval and add half the width.

$$x_0^* = x_0 + \frac{1}{6} = \frac{7}{6} \qquad x_1^* = x_1 + \frac{1}{6} = \frac{3}{2} \qquad x_2^* = x_2 + \frac{1}{6} = \frac{11}{6}$$

$$x_3^* = x_3 + \frac{1}{6} = \frac{13}{6} \qquad x_4^* = x_4 + \frac{1}{6} = \frac{5}{2} \qquad x_5^* = x_5 + \frac{1}{6} = \frac{17}{6}$$

Since every rectangle has the same width, we know $\Delta x_j = \frac{1}{3}$ for all indices $0 \le j < 6$. Therefore,

$$\int_1^3 2x\cos(x)dx \approx \sum_{j=0}^{5} 2x_j^*\cos(x_j^*)\,\Delta x_j$$

$$= 2x_0^*\cos(x_0^*)\Delta x_0 + 2x_1^*\cos(x_1^*)\Delta x_1 + 2x_2^*\cos(x_2^*)\Delta x_2 + 2x_3^*\cos(x_3^*)\Delta x_3 + 2x_4^*\cos(x_4^*)\Delta x_4 + 2x_5^*\cos(x_5^*)\Delta x_5$$

$$= \frac{1}{3}\left[2\cdot\frac{7}{6}\cos\left(\frac{7\pi}{6}\right) + 2\cdot\frac{3}{2}\cos\left(\frac{3\pi}{2}\right) + 2\cdot\frac{11}{6}\cos\left(\frac{11\pi}{6}\right) + 2\cdot\frac{13}{6}\cos\left(\frac{13\pi}{6}\right) + 2\cdot\frac{5}{2}\cos\left(\frac{5\pi}{2}\right) + 2\cdot\frac{17}{6}\cos\left(\frac{17\pi}{6}\right)\right]$$

$$\approx 2.694$$

Problem 7. Use a midpoint approximation with five approximating rectangles to estimate the value of

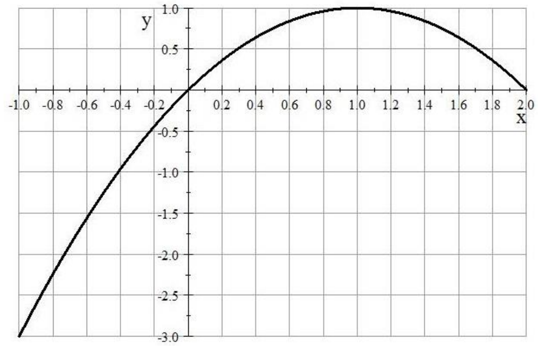
$$\int_{-1}^{2} [1 - (x-1)^2]\,dx$$

Part (a). What is the width of each rectangle used in this approximation?

Part (b). What are the six numbers in the partition? Use proper notation.

Part (c). What are the five tags? Use proper notation.

Part (d). The diagram below shows the graph of the function $y = f(x) = 1 - (x-1)^2$. Draw your approximating rectangles on the diagram.

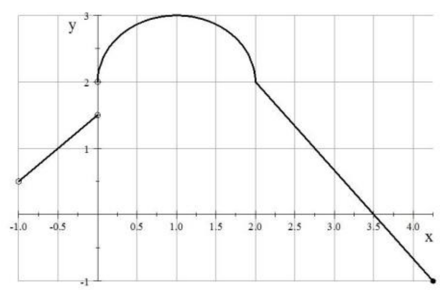


Part (e). Construct the Riemann Sum that approximates the net area and then compute your approximation.

$$\int_{-1}^{2} [1-(x-1)^2]dx \approx$$

Homework.

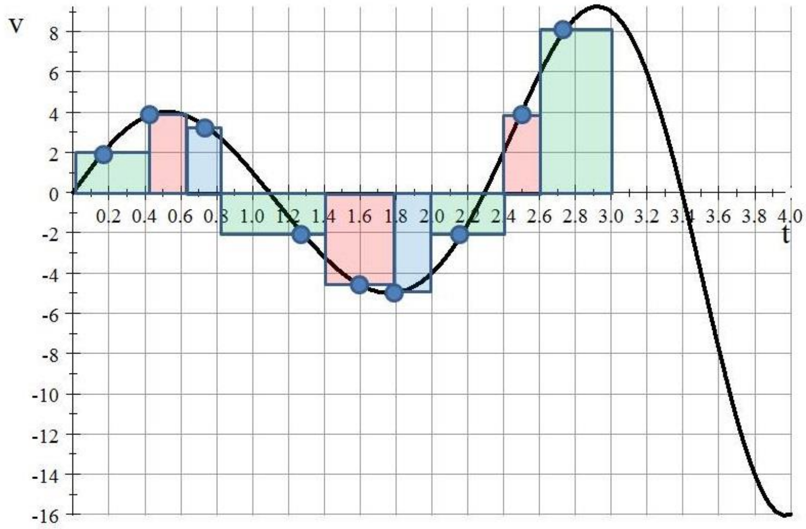
The diagram below shows the graph of a function $y = f(x)$. Use this graph to answer Problems 1 – 6. The arc represents a semicircle of radius 1 centered at (1,2).



(1) $\int_{0}^{2} f(x)\,dx$ (2) $\int_{1}^{3.5} f(x)\,dx$ (3) $\int_{2}^{4.5} f(x)\,dx$

(4) $\int_{-0.5}^{1} f(x)\,dx$ (5) $\int_{-1}^{2} f(x)\,dx$ (6) $\int_{-1}^{4.5} f(x)\,dx$

The diagram below shows the graph of a function $v = f(t)$ along with some approximating rectangles on the input interval $0 \leq t \leq 3$. This diagram is used in Problems 7 – 11.



Problem 7. The bases of the approximating rectangles shown above serve to partition the input interval $0 \leq t \leq 3$. Write down the numbers in this partition.

$t_0 =$ ________ $t_1 =$ ________ $t_2 =$ ________ $t_3 =$ ________ $t_4 =$ ________

$t_5 =$ ________ $t_6 =$ ________ $t_7 =$ ________ $t_8 =$ ________ $t_9 =$ ________

Problem 8. Each pair of consecutive numbers in the partition forms the endpoints of a subinterval used as the base of a rectangle. A tag was selected from each of these subintervals, and the corresponding output from f was used as the "signed" height of the rectangle. Identify the tag selected from each subinterval.

$t_0^* =$ ________ $t_1^* =$ ________ $t_2^* =$ ________ $t_3^* =$ ________ $t_4^* =$ ________

$t_5^* =$ ________ $t_6^* =$ ________ $t_7^* =$ ________ $t_8^* =$ ________

Problem 9. What is the width of each of the approximating rectangles shown above?

$\Delta t_0 =$ ________ $\Delta t_1 =$ ________ $\Delta t_2 =$ ________ $\Delta t_3 =$ ________

$\Delta t_4 =$ ________ $\Delta t_5 =$ ________ $\Delta t_6 =$ ________ $\Delta t_7 =$ ________

$\Delta t_8 =$ ________

Problem 10. Use the approximating rectangles shown to estimate the value of the net area.

$\int_0^3 f(t)dt \approx$

Problem 11. In this problem, you will construct a midpoint estimate for the net area $\int_0^3 f(t)dt$.

Part (a). Suppose we divide the input interval $0 \leq t \leq 3$ into subintervals that each has width 0.2. How many of these subintervals will be required?

Part (b). The endpoints of these subintervals constitute a partition of the input interval $0 \leq t \leq 3$. Write down the elements of this partition.

$t_0 =$ ________ $t_1 =$ ________ $t_2 =$ ________ $t_3 =$ ________ $t_4 =$ ________

$t_5 =$ ________ $t_6 =$ ________ $t_7 =$ ________ $t_8 =$ ________ $t_9 =$ ________

$t_{10} =$ ________ $t_{11} =$ ________ $t_{12} =$ ________ $t_{13} =$ ________

$t_{14} =$ ________ $t_{15} =$ ________

Part (c). Write down the midpoint of each of these subintervals.

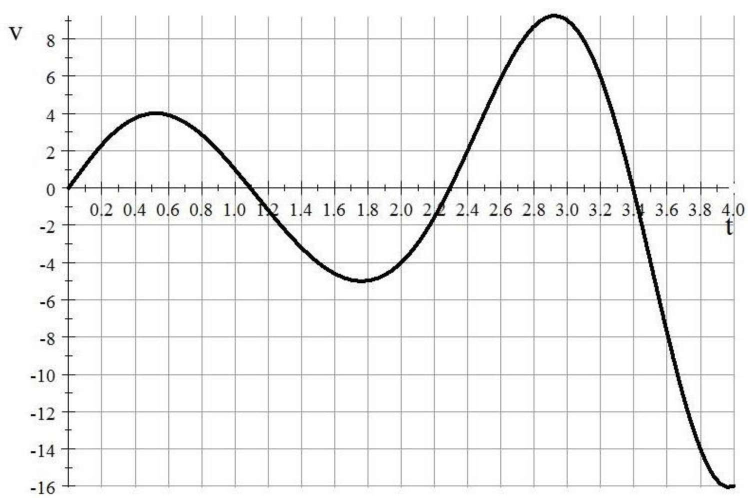
$t_0^* =$ ________ $t_1^* =$ ________ $t_2^* =$ ________ $t_3^* =$ ________ $t_4^* =$ ________

$t_5^* =$ ________ $t_6^* =$ ________ $t_7^* =$ ________ $t_8^* =$ ________ $t_9^* =$ ________

$t_{10}^* =$ ________ $t_{11}^* =$ ________ $t_{12}^* =$ ________ $t_{13}^* =$ ________ $t_{14}^* =$ ________

Part (d). Estimate the output of the function f at each of the numbers you wrote in Part (c). Using these estimates as the "signed" heights, draw the approximating rectangles on the diagram below.

$f(t_0^*) \approx$	$f(t_1^*) \approx$
$f(t_2^*) \approx$	$f(t_3^*) \approx$
$f(t_4^*) \approx$	$f(t_5^*) \approx$
$f(t_6^*) \approx$	$f(t_7^*) \approx$
$f(t_8^*) \approx$	$f(t_9^*) \approx$
$f(t_{10}^*) \approx$	$f(t_{11}^*) \approx$
$f(t_{12}^*) \approx$	$f(t_{13}^*) \approx$
$f(t_{14}^*) \approx$	



Part (e). Construct the midpoint estimate based on these approximating rectangles.

$$\int_0^3 f(t)dt \approx$$

Problem 12. Write down the Riemann sum for the midpoint estimate that uses six rectangles to approximate

$$\int_1^3 \sqrt{x}\, dx$$

Problem 13. Write down the Riemann sum for the midpoint estimate that uses eight rectangles to approximate

$$\int_{-1}^1 x\sin(x)\, dx$$

Problem 14. Write down the Riemann sum for the midpoint estimate that uses ten rectangles to approximate

$$\int_2^5 \frac{1}{x} dx$$

Answers to the Homework.

(1) $\int_0^2 f(x)\, dx = 4 + \frac{\pi}{2}$ (2) $\int_1^{3.5} f(x)\, dx = 3.5 + \frac{\pi}{4}$ (3) $\int_2^{4.5} f(x)\, dx = 1$

(4) $\int_{-0.5}^1 f(x)\, dx = \frac{11+\pi}{4}$ (5) $\int_{-1}^2 f(x)\, dx = 5 + \frac{\pi}{2}$ (6) $\int_{-1}^{4.5} f(x)\, dx = 6 + \frac{\pi}{2}$

Problem 7.

$t_0 = \mathbf{0}$ $t_1 = \mathbf{0.41}$ $t_2 = \mathbf{0.62}$ $t_3 = \mathbf{0.81}$ $t_4 = \mathbf{1.4}$

$t_5 = \mathbf{1.8}$ $t_6 = \mathbf{2.0}$ $t_7 = \mathbf{2.4}$ $t_8 = \mathbf{2.6}$ $t_9 = \mathbf{3.0}$

Problem 8.

$t_0^* = \mathbf{0.17}$ $t_1^* = \mathbf{0.41}$ $t_2^* = \mathbf{0.75}$ $t_3^* = \mathbf{1.25}$ $t_4^* = \mathbf{1.6}$

$t_5^* = \mathbf{1.8}$ $t_6^* = \mathbf{2.17}$ $t_7^* = \mathbf{2.5}$ $t_8^* = \mathbf{2.7}$

Problem 9.

$\Delta t_0 = \mathbf{0.41}$ $\Delta t_1 = \mathbf{0.21}$ $\Delta t_2 = \mathbf{0.19}$ $\Delta t_3 = \mathbf{0.59}$

$\Delta t_4 = \mathbf{0.2}$ $\Delta t_5 = \mathbf{0.2}$ $\Delta t_6 = \mathbf{0.4}$ $\Delta t_7 = \mathbf{0.2}$

$\Delta t_8 = \mathbf{0.4}$

Problem 10.

$$\int_0^3 f(t)dt \approx f(t_0^*)\Delta t_0 + f(t_1^*)\Delta t_1 + f(t_2^*)\Delta t_2 + f(t_3^*)\Delta t_3 + f(t_4^*)\Delta t_4 + f(t_5^*)\Delta t_5 + f(t_6^*)\Delta t_6 + f(t_7^*)\Delta t_7 + f(t_7^*)\Delta t_7$$

$$\approx 2 \cdot 0.41 + 3.9 \cdot 0.21 + 3 \cdot 0.19 - 2 \cdot 0.59 - 4.5 \cdot 0.2 - 5 \cdot 0.2 - 2 \cdot 0.4 + 3.9 \cdot 0.2 + 8 \cdot 0.4$$

$$= 2.309$$

Problem 11.

Part (a). There will be $n = \frac{3}{0.2} = 15$ subinterinvals.

Part (b).

$t_0 = \mathbf{0}$ $\quad t_1 = \mathbf{0.2}$ $\quad t_2 = \mathbf{0.4}$ $\quad t_3 = \mathbf{0.6}$ $\quad t_4 = \mathbf{0.8}$

$t_5 = \mathbf{1}$ $\quad t_6 = \mathbf{1.2}$ $\quad t_7 = \mathbf{1.4}$ $\quad t_8 = \mathbf{1.6}$ $\quad t_9 = \mathbf{1.8}$

$t_{10} = \mathbf{2}$ $\quad t_{11} = \mathbf{2.2}$ $\quad t_{12} = \mathbf{2.4}$ $\quad t_{13} = \mathbf{2.6}$

$t_{14} = \mathbf{2.8}$ $\quad t_{15} = \mathbf{3}$

Part (c).

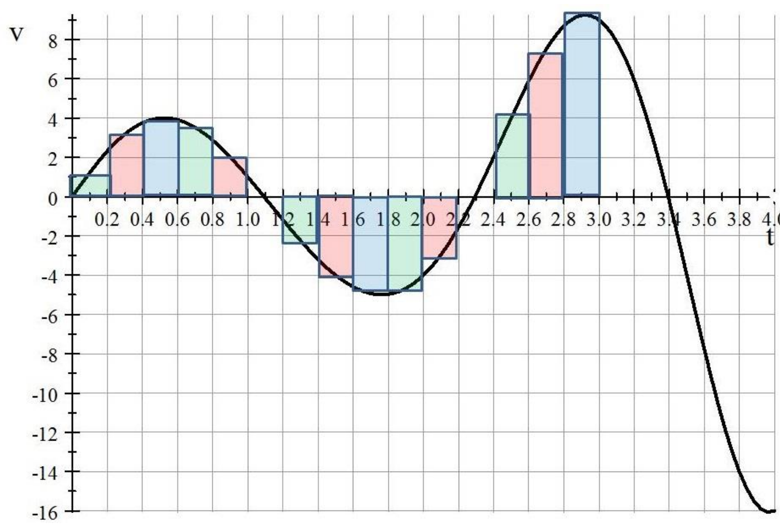
$t_0^* = \mathbf{0.1}$ $\quad t_1^* = \mathbf{0.3}$ $\quad t_2^* = \mathbf{0.5}$ $\quad t_3^* = \mathbf{0.7}$ $\quad t_4^* = \mathbf{0.9}$

$t_5^* = \mathbf{1.1}$ $\quad t_6^* = \mathbf{1.3}$ $\quad t_7^* = \mathbf{1.5}$ $\quad t_8^* = \mathbf{1.7}$ $\quad t_9^* = \mathbf{1.9}$

$t_{10}^* = \mathbf{2.1}$ $\quad t_{11}^* = \mathbf{2.3}$ $\quad t_{12}^* = \mathbf{2.5}$ $\quad t_{13}^* = \mathbf{2.7}$ $\quad t_{14}^* = \mathbf{2.9}$

Part (d).

$f(t_0^*) \approx \mathbf{1.0}$	$f(t_1^*) \approx \mathbf{2.5}$
$f(t_2^*) \approx \mathbf{4.0}$	$f(t_3^*) \approx \mathbf{3.5}$
$f(t_4^*) \approx \mathbf{2.5}$	$f(t_5^*) \approx \mathbf{0.0}$
$f(t_6^*) \approx \mathbf{-2.0}$	$f(t_7^*) \approx \mathbf{-4.0}$
$f(t_8^*) \approx \mathbf{-5.0}$	$f(t_9^*) \approx \mathbf{-4.75}$
$f(t_{10}^*) \approx \mathbf{-3.0}$	$f(t_{11}^*) \approx \mathbf{0.0}$
$f(t_{12}^*) \approx \mathbf{4.0}$	$f(t_{13}^*) \approx \mathbf{7.0}$
$f(t_{14}^*) \approx \mathbf{9.0}$	



Part (e). Construct the midpoint estimate based on these approximating rectangles.

$$\int_0^3 f(t)dt \approx 0.2(1 + 2.5 + 4.0 + 3.5 + 2.5 + 0.0 - 2.0 - 4.0 - 5.0 - 4.75 - 3.0 + 0.0 + 4.0 + 7.0 + 9.0)$$

$$= 2.95$$

Problem 12. First, note that the width of each rectangle will be $(3-1)/6 = 1/3$. Therefore, the partition we use to create the rectangles will be

$$x_0 = 1 \quad x_1 = \frac{4}{3} \quad x_2 = \frac{5}{3} \quad x_3 = 2 \quad x_4 = \frac{7}{3} \quad x_5 = \frac{9}{3} \quad x_6 = 3$$

The tags used in the approximation will be the midpoint of each subinterval created by consecutive numbers in the partition. In particular,

$$x_0^* = \frac{7}{6} \quad x_1^* = \frac{3}{2} \quad x_2^* = \frac{11}{6} \quad x_3^* = \frac{13}{6} \quad x_4^* = \frac{5}{2} \quad x_5^* = \frac{17}{6}$$

The midpoint approximation will be

$$\int_1^3 \sqrt{x}\,dx \approx \sum_{j=0}^{5} \sqrt{x_j^*}\ \Delta x_j$$

$$= \sqrt{\frac{7}{6}}\cdot\left(\frac{1}{3}\right) + \sqrt{\frac{3}{2}}\cdot\left(\frac{1}{3}\right) + \sqrt{\frac{11}{6}}\cdot\left(\frac{1}{3}\right) + \sqrt{\frac{13}{6}}\cdot\left(\frac{1}{3}\right) + \sqrt{\frac{5}{2}}\cdot\left(\frac{1}{3}\right) + \sqrt{\frac{17}{6}}\cdot\left(\frac{1}{3}\right)$$

$$\approx 2.7984$$

Problem 13. First, note that the width of each rectangle will be $(1+1)/8 = 1/4$. Therefore, the partition we use to create the rectangles will be

$$x_0 = -1 \quad x_1 = -\frac{3}{4} \quad x_2 = -\frac{1}{2} \quad x_3 = -\frac{1}{4} \quad x_4 = 0 \quad x_5 = \frac{1}{4} \quad x_6 = \frac{1}{2} \quad x_7 = \frac{3}{4} \quad x_8 = 1$$

The tags used in the approximation will be the midpoint of each subinterval created by consecutive numbers in the partition. In particular,

$$x_0^* = -\frac{7}{8} \quad x_1^* = -\frac{5}{8} \quad x_2^* = -\frac{3}{8} \quad x_3^* = -\frac{1}{8} \quad x_4^* = \frac{1}{8} \quad x_5^* = \frac{3}{8} \quad x_6^* = \frac{5}{8} \quad x_7^* = \frac{7}{8}$$

The midpoint approximation will be

$$\int_{-1}^{1} x\sin(x)\,dx \approx \sum_{j=0}^{7} x_j^*\sin(x_j^*)\ \Delta x_j$$

$$= \left(-\frac{7}{8}\right)\sin\left(-\frac{7}{8}\right)\cdot\left(\frac{1}{4}\right) + \left(-\frac{5}{8}\right)\sin\left(-\frac{5}{8}\right)\cdot\left(\frac{1}{4}\right) + \left(-\frac{3}{8}\right)\sin\left(-\frac{3}{8}\right)\cdot\left(\frac{1}{4}\right) + \left(-\frac{1}{8}\right)\sin\left(-\frac{1}{8}\right)\cdot\left(\frac{1}{4}\right) + \left(\frac{1}{8}\right)\sin\left(\frac{1}{8}\right)\cdot\left(\frac{1}{4}\right)$$
$$+ \left(\frac{3}{8}\right)\sin\left(\frac{3}{8}\right)\cdot\left(\frac{1}{4}\right) + \left(\frac{5}{8}\right)\sin\left(\frac{5}{8}\right)\cdot\left(\frac{1}{4}\right) + \left(\frac{7}{8}\right)\sin\left(\frac{7}{8}\right)\cdot\left(\frac{1}{4}\right)$$

$$\approx 0.29287$$

Problem 14. First, note that the width of each rectangle will be $(5-2)/10 = 3/10$. Therefore, the partition we use to create the rectangles will be

$$x_0 = 2 \quad x_1 = \frac{23}{10} \quad x_2 = \frac{13}{5} \quad x_3 = \frac{29}{10} \quad x_4 = \frac{16}{5} \quad x_5 = \frac{7}{2} \quad x_6 = \frac{19}{5} \quad x_7 = \frac{41}{10} \quad x_8 = \frac{22}{5} \quad x_9 = \frac{47}{10} \quad x_{10} = 5$$

The tags used in the approximation will be the midpoint of each subinterval created by consecutive numbers in the partition. In particular,

$$x_0^* = \frac{43}{20} \quad x_1^* = \frac{49}{20} \quad x_2^* = \frac{11}{4} \quad x_3^* = \frac{61}{20} \quad x_4^* = \frac{67}{20} \quad x_5^* = \frac{73}{20} \quad x_6^* = \frac{79}{20} \quad x_7^* = \frac{17}{4} \quad x_8^* = \frac{91}{20} \quad x_9^* = \frac{97}{20}$$

The midpoint approximation will be

$$\int_2^5 \frac{1}{x}\,dx \approx \sum_{j=0}^{9} \frac{1}{x_j^*}\,\Delta x_j$$

$$= \frac{20}{43}\cdot\left(\frac{3}{10}\right) + \frac{20}{49}\cdot\left(\frac{3}{10}\right) + \frac{4}{11}\cdot\left(\frac{3}{10}\right) + \frac{20}{61}\cdot\left(\frac{3}{10}\right) + \frac{20}{67}\cdot\left(\frac{3}{10}\right) + \frac{20}{73}\cdot\left(\frac{3}{10}\right) + \frac{20}{79}\cdot\left(\frac{3}{10}\right) + \frac{4}{17}\cdot\left(\frac{3}{10}\right) + \frac{20}{91}\cdot\left(\frac{3}{10}\right) + \frac{20}{97}\cdot\left(\frac{3}{10}\right)$$

$$\approx 0.9155$$