In the previous investigation, we introduced the notion of net area enclosed between the graph of an integrable function and the input axis. In this investigation, we take the exploration a step further.

Let $y = f(x)$ be a function that is integrable on an input interval $L \leq x \leq U$. For any positive integer n, we can construct a partition

$$x_0 = L \quad x_1 \quad x_2 \quad \ldots \quad x_{n-1} \quad x_n = U$$

and select tags $x_0^* \quad x_1^* \quad \ldots \quad x_{n-1}^*$ from the subintervals formed by consecutive numbers in the partition. Once we have selected our partition, there will always be a subinterval whose width is the largest of all the subintervals we have chosen. It is customary to let $\text{Max}(\Delta x)$ represent this largest subinterval width.

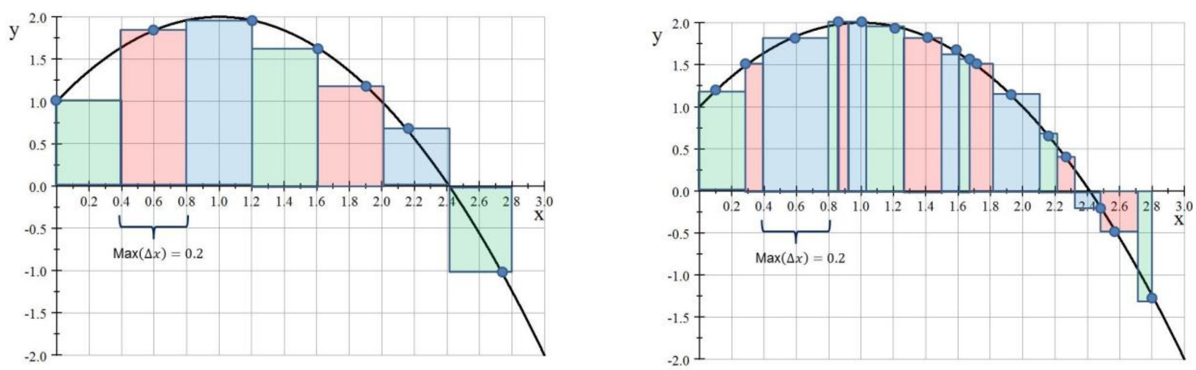
Net Area for an Integrable Function

Let $y = f(x)$ be an integrable function on the input interval $L \leq x \leq U$. The *net area* enclosed between the graph of the function f and the input axis on the input interval $L \leq x \leq U$ is defined by the limiting process

$$\int_L^U f(x)\,dx = \lim_{\text{Max}(\Delta x)\to 0} \sum_{j=0}^{n-1} f\left(x_j^*\right)\Delta x_j$$

By considering the end result of the limiting process that results by forcing $\text{Max}(\Delta x)$ to become smaller and smaller, we are effectively forcing the approximating rectangles in the Riemann sums to become ever more numerous and thinner. This in turn forces the Riemann sums to become ever better approximations to the net area.

As an example, let's consider the function $f(x) = 2 - (x-1)^2$ on the input interval $0 \leq x \leq 2.8$. The diagram below shows two families of approximating rectangles for the function on this input interval.



In both cases, the largest subinterval in the partition has width 0.2 units. In the partition on the left, *every* subinterval has the same width, but in the partition on the right, only one subinterval has the maximum allowed width. There are infinitely many ways we could construct a partition of the input interval whose maximum allowed subinterval width is 0.2 units; however, it seems reasonable that *all* of them would yield a better approximation to the net area than would a family of rectangles arising from a partition in which $\text{Max}(\Delta x) = 0.5$.

The limit definition of net area is seldom used to compute actual net areas. It is used primarily in mathematical modeling (as you will see in Calculus II). It is also used to prove one of the most amazing results in all of mathematics.

Before we see how the limit definition can be used to prove this result, let's take a look at the result itself.

Problem 1. In the previous homework assignment, you used a midpoint approximation to show that

$$\int_2^5 \frac{1}{x}\,dx \approx 0.9155$$

Compare this estimate to the value of $\ln(5) - \ln(2)$. What do you notice?

Problem 2. In the previous homework assignment, you used a midpoint approximation to show that

$$\int_1^3 \sqrt{x}\,dx \approx 2.7984$$

Compare this estimate to the value of $\frac{2}{3}(3)^{3/2} - \frac{2}{3}(1)^{3/2}$. What do you notice?

Net Area Theorem (Second Fundamental Theorem of Calculus)

Suppose that $r = f(x)$ is continuous on the input interval $L \le x \le U$. If $y = F(x)$ is any antiderivative for the function f, then

$$\int_L^U f(x)\,dx = F(U) - F(L)$$

Problem 3. Consider the function $f(x) = \sqrt[3]{x^4}$.

Part (a). We know that $F(x) = \frac{3}{7}\sqrt[3]{x^7} + 1$ is one antiderivative for the function f. Use this antiderivative to determine the value of

$$\int_{-1}^3 \sqrt[3]{x^4}\,dx$$

Part (b). We know that $G(x) = \frac{3}{7}\sqrt[3]{x^7} - 2$ is one antiderivative for the function f. Use this antiderivative to determine the value of

$$\int_{-1}^{3} \sqrt[3]{x^4}\, dx$$

Problem 4. Construct an antiderivative for the function $f(x) = 2\cos(x)$ and use it to determine the value of

$$\int_{0}^{4} 2\cos(x)\, dx$$

Problem 5. Construct an antiderivative for the function $f(x) = 4 + e^x$ and use it to determine the value of

$$\int_{2}^{3} (4 + e^x)\, dx$$

Problem 6. Use the Net Area Theorem to show that

$$\int_{3}^{1} \left(4\sqrt{t} - 2t\right)dt = \frac{8}{3}\left(1 - \sqrt{27}\right) + 8$$

Example 2. Use the Net Area Theorem to determine the exact value of

$$\int_2^4 \frac{r}{3+r^2}\,dr$$

Solution. First, we need to identify one antiderivative for the function

$$y = f(r) = \frac{r}{3+r^2}$$

To this end, observe that the antiderivative family for this function is given by

$$\int \frac{r}{3+r^2}\,dr = \int \frac{1}{3+r^2}[r]\,dr \qquad \text{Let } u = 3+r^2 \text{ so } \frac{du}{dr} = 2r$$

$$= \int \frac{1}{u}\left[\frac{1}{2}\cdot\frac{du}{dr}\right]dr$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|3+r^2| + C$$

The Net Area Theorem tells us that we may use any antiderivative for the function f. This means we are free to choose whatever value of C we want. For example, let's consider the antiderivative

$$F(r) = \frac{1}{2}\ln|3+r^2| - 7$$

If we use this antiderivative, then

$$\int_2^4 \frac{r}{3+r^2}\,dr = F(4) - F(2) = \left[\frac{1}{2}\ln|3+16| - 7\right] - \left[\frac{1}{2}\ln|3+4| - 7\right] = \frac{1}{2}[\ln(19) - \ln(7)]$$

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Notice that our choice of $C = -7$ in the previous example did not affect the final answer to the problem, since the two C-terms merely cancelled each other. Because of this, it is customary simply to let $C = 0$.

Problem 6. Construct an antiderivative for the function $f(x) = x\sin(x^2)$ and use it to determine the value of

$$\int_0^4 x\sin(x^2)\,dx$$

Problem 4. Use the Net Area Theorem to evaluate the following.

$$\int_{-1}^{4} (3v^2 - 2v) dv \qquad \int_{\pi}^{0} (\sin(x) + \cos(x))\, dx$$

Problem 5 . Use the Net Area Theorem to evaluate the following.

$$\int_{0}^{3} 2t(2 + t^2)^3\, dt$$

It is worth explaining why the Net Area Theorem is true. To begin, let n be any positive integer, and construct any partition of the input interval $L \leq x \leq U$. The numbers appearing in this partition will be

$$x_0 = L \qquad x_1 \qquad x_2 \qquad \dots \qquad x_{n-1} \qquad x_n = U$$

Now, consider any two consecutive numbers in this partition. Let's call them x_j and x_{j+1} and think about the function F on the input interval $x_j \leq x \leq x_{j+1}$. First, note that the width of this input interval is

$$\Delta x = x_{j+1} - x_j$$

We know that the function F is differentiable on the input interval $x_j \leq x \leq x_{j+1}$. (We have assumed that $F' = f$.) Now, we will play a little algebra trick. Observe

$$F(b) - F(a) = F(b) + F(x_{n-1}) - F(x_{n-1}) + \cdots + F(x_1) - F(x_1) - F(a)$$

$$= \left[\frac{F(b) - F(x_{n-1})}{\Delta x_{n-1}}\right] \cdot \Delta x_{n-1} + \left[\frac{F(x_{n-1}) - F(x_{n-2})}{\Delta x_{n-2}}\right] \cdot \Delta x_{n-2} + \cdots + \left[\frac{F(x_1) - F(a)}{\Delta x_0}\right] \cdot \Delta x_0$$

Now, the term

$$\frac{F\left(x_j + h\right) - F\left(x_j\right)}{\Delta x_j}$$

represents the *average rate of change* for the function F on the input interval $x_j \le x \le x_{j+1}$. Consequently, the Mean Value Theorem tells us there exists some input value x_j^* in this interval where

$$F'\left(x_j^*\right) = \frac{F\left(x_j + h\right) - F\left(x_j\right)}{\Delta x_j}$$

We have assumed that $F'\left(x_j^*\right) = f\left(x_j^*\right)$. Therefore, we know

$$F(U) - F(L) = F(U) + F(x_{n-1}) - F(x_{n-1}) + \cdots + F(x_1) - F(x_1) - F(L)$$

$$= \left[\frac{F(U) - F(x_{n-1})}{\Delta x_{n-1}}\right] \cdot \Delta x_{n-1} + \left[\frac{F(x_{n-1}) - F(x_{n-2})}{\Delta x_{n-2}}\right] \cdot \Delta x_{n-2} + \cdots + \left[\frac{F(x_1) - F(L)}{\Delta x_0}\right] \cdot \Delta x_0$$

$$= f(x_{n-1}^*)\Delta x_{n-1} + f(x_{n-2}^*)\Delta x_{n-2} + \cdots + f(x_0^*)\Delta x_0$$

$$= \sum_{j=0}^{n-1} f\left(x_j^*\right)\Delta x_j$$

We may therefore conclude

$$F(U) - F(L) = \lim_{\text{Max}(\Delta x)\to 0}[F(U) - F(L)] = \lim_{\text{Max}(\Delta x)\to 0}\sum_{j=0}^{n-1} f\left(x_j^*\right)\Delta x_j = \int_L^U f(x)\,dx$$

Homework for Investigation 17

Use the First Fundamental Theorem of Calculus to find the exact values for the following.

(1) $\int_1^3 \ln(x)\,dx$ (2) $\int_1^4 \left(\frac{1}{\sqrt{x}} - \frac{2}{\sqrt[3]{x}}\right) dx$ (3) $\int_0^{5/2} \cos(\pi x)\,dx$ (4) $\int_{-1}^2 e^{-x}\,dx$

(5) $\int_2^6 (2x - 6x^2 + 5)\,dx$ (6) $\int_{-3}^2 x\sqrt[5]{3 - x^2}\,dx$ (7) $\int_0^{10} \frac{x}{1 + x^2}\,dx$ (8) $\int_1^4 \sin^3(\pi x)\cos(\pi x)\,dx$

Answers.

Problem 1. Since one antiderivative for $f(x) = \ln(x)$ is the function $F(x) = x\ln(x) - x$, we know

$$\int_1^3 \ln(x)\,dx = F(3) - F(1) = [3\ln(3) - 3] - [1\ln(1) - 1] = 3\ln(3) - 2$$

Problem 2. Observe that

$$\int \left(\frac{1}{\sqrt{x}} - \frac{2}{\sqrt[3]{x}}\right) dx = \int x^{-1/2}\,dx - 2\int x^{-1/3}\,dx = 2\sqrt{x} - 3\sqrt[3]{x^2} + C$$

Therefore, we know

$$\int_{-1}^4 \left(\frac{1}{\sqrt{x}} - \frac{2}{\sqrt[3]{x}}\right) dx = \left[2\sqrt{4} - 3\sqrt[3]{16}\right] - \left[2\sqrt{1} - 3\sqrt[3]{1}\right] = 5 - 6\sqrt[3]{2}$$

Problem 3. Observe that

$$\int \cos(\pi x)\,dx = \int \cos(\pi x)[1]\,dx \qquad \text{Let } u = \pi x \text{ so that } \frac{1}{\pi}\cdot\frac{du}{dx} = 1$$

$$= \int \cos(u)\left[\frac{1}{\pi}\cdot\frac{du}{dx}\right] dx$$

$$= \frac{1}{\pi}\sin(\pi x) + C$$

Therefore, we know

$$\int_0^{5/2} \cos(\pi x)\,dx = \left[\frac{1}{\pi}\sin\left(\frac{5\pi}{2}\right)\right] - \left[\frac{1}{\pi}\sin(0)\right] = -\frac{1}{\pi}$$

Problem 4. Observe that

$$\int e^{-x}\,dx = \int e^{-x}[1]\,dx \qquad \text{Let } u = -x \text{ so that } (-1)\cdot\frac{du}{dx} = 1$$

$$= \int e^{u}\left[(-1)\cdot\frac{du}{dx}\right]dx$$

$$= -e^{-x} + C$$

Therefore, we know

$$\int_{-1}^{2} e^{-x}\,dx = [-e^{-2}] - [-e^{1}] = e - \frac{1}{e^2}$$

Problem 5. Observe that

$$\int (2x - 6x^2 + 5)dx = 2\int x\,dx - 6\int x^2\,dx + \int 5\,dx = x^2 - 2x^3 + 5x + C$$

Therefore, we know

$$\int_{2}^{6} (2x - 6x^2 + 5)\,dx = [6^2 - 2(6)^3 + 5(6)] - [2^2 - 2(2)^3 + 5(2)] = 32$$

Problem 6. Observe that

$$\int x\sqrt[5]{3 - x^2}\,dx = \int (3 - x^2)^{1/5}[x]\,dx \qquad \text{Let } u = 3 - x^2 \text{ so that } -\frac{1}{2}\cdot\frac{du}{dx} = x$$

$$= \int u^{1/5}\left[-\frac{1}{2}\cdot\frac{du}{dx}\right]dx$$

$$= -\frac{5}{12}(3 - x^2)^{6/5} + C$$

Therefore, we know

$$\int_{-3}^{2} x\sqrt[5]{3 - x^2}\,dx = \left[-\frac{5}{12}(3 - 2^2)^{6/5}\right] - \left[-\frac{5}{12}(3 - (-3)^2)^{6/5}\right] = \frac{5}{12}\left(6\sqrt[5]{6} - 1\right)$$

Problem 7. Observe that

$$\int \frac{x}{1 + x^2}\,dx = \int \frac{1}{1 + x^2}[x]\,dx \qquad \text{Let } u = 1 + x^2 \text{ so that } \frac{1}{2}\cdot\frac{du}{dx} = x$$

$$= \int \frac{1}{u}\left[\frac{1}{2}\cdot\frac{du}{dx}\right]dx$$

$$= \frac{1}{2}\ln|1 + x^2| + C$$

Therefore, we know

$$\int_{0}^{10} \frac{x}{1 + x^2}\,dx = \left[\frac{1}{2}\ln|1 + 10^2|\right] - \left[\frac{1}{2}\ln|1 + 0^2|\right] = \frac{1}{2}\ln(101)$$

Problem 8. Observe that

$$\int \sin^3(\pi x)\cos(\pi x)\,dx = \int (\sin(\pi x))^3[\cos(\pi x)]\,dx \qquad \text{Let } u = \sin(\pi x) \text{ so that } \frac{1}{\pi}\cdot\frac{du}{dx} = \cos(x)$$

$$= \int u^3\left[\frac{1}{\pi}\cdot\frac{du}{dx}\right]dx$$

$$= \frac{1}{4\pi}(\sin(\pi x))^4 + C$$

Therefore, we know

$$\int_1^4 \sin^3(\pi x)\cos(\pi x)\,dx = \left[\frac{1}{4\pi}(\sin(4\pi))^4\right] - \left[\frac{1}{4\pi}(\sin(\pi))^4\right] = 0$$