In this investigation, we will review one of the most important concepts used in calculus --- the notion of *linear* relationships between covarying quantities.

Suppose that x represents the values of some defined quantity. The *change* in the measure of the quantity *from* a value $x = a$ *to* a value $x = b$ is denoted by

$$\Delta x = b - a$$

Constant Rate of Change

Suppose that x and y represent the values of two covarying quantities. We say that the two quantities *change at a constant rate* with respect to each other provided a change in the values of x is always proportional to the corresponding change in the values of y. In other words, there is a constant m such that

$$\Delta y = m\Delta x$$

When this is true, we way that the values of x and y are *linearly related.* The constant m is called the *rate of change of* y *with respect to* x.

Problem 1. Suppose Kim is riding her bike along a straight road at a constant rate of 0.56 km/min. Kim passes a coffee shop while traveling at this constant rate. At 9:30 AM, Kim is 3 km past the coffee shop.

Part (a). How far is Kim from the coffee shop at 9:31 AM?

Part (b). How far is Kim from the coffee shop 24.6 minutes past 9:30 AM?

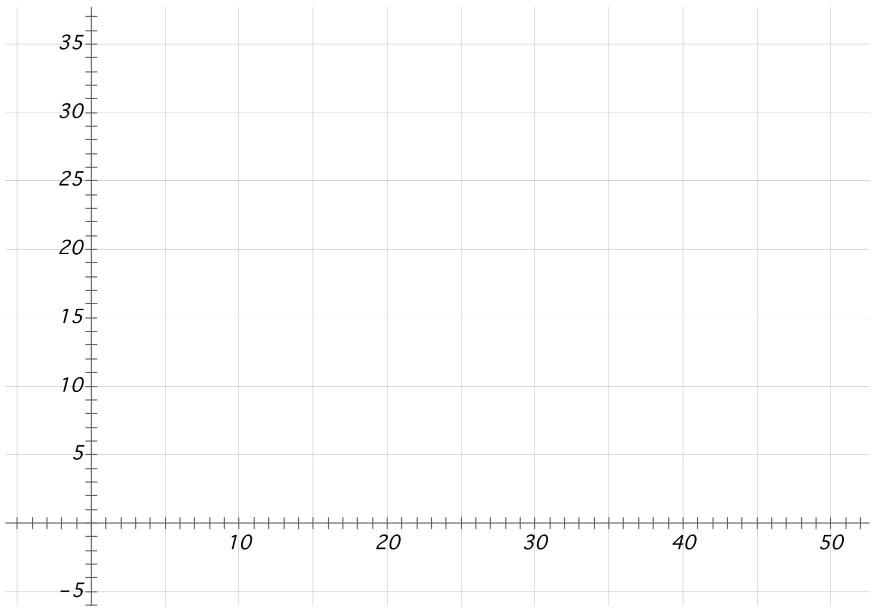
Part (c). Suppose we let t represent the values of the quantity "The time passed since Kim was three kilometers past the coffee shop," measured in minutes. What quantity in this process covaries with the time quantity? Assign a variable name to this quantity and be sure to properly identify it.

Part (d). Define a formula that relates the measures of your quantity to the measures of the time quantity.

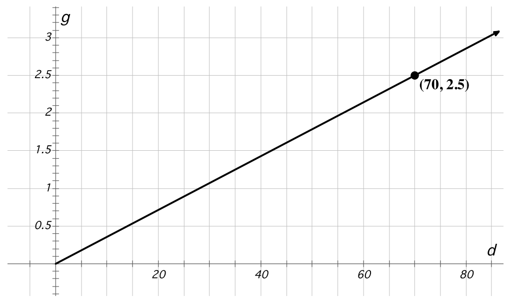
Part (e). Define a formula that relates the *change* in the values of t to the corresponding *change* in the values of your variable. How is this formula different from the one you created in Part (d)?

Part (f). Use your response to Part (e) to determine what time Kim passed the coffee shop.

Part (g). Sketch a graph of the relationship between the values of t and the values of your variable. Be sure to label your axes.



Problem 2. John Paul is driving on Interstate 35 from Norman, OK to Stillwater, OK. John Paul's car consumes fuel at a constant rate while he drives on I-35. The graph below represents the relationship between the number of miles John Paul has driven on I-35 (represented by the variable *d*) and the number of gallons of fuel his car has consumed since he started driving on I-35 (represented by the variable *g*). The point (70, 2.5) is on the graph, as indicated.



Part (a). As the number of miles that John Paul has driven on I-35 changes from 0 to 12 miles, how much does the amount of fuel his car has consumed change? Represent this change on the graph and explain how you determined this change.

Part (b). As the number of miles that John Paul has driven on I-35 changes from 21 to 27 miles, how much does the amount of fuel his car has consumed change? Represent this change on the graph and explain how you determined this change.

Part (c). How much does the amount of fuel John Paul's car consumed change for any change of 6 miles he has driven on I-35?

Part (d). How are Δd and Δg related? Write a formula that expresses the relationship between Δd and Δg.

Part (e). Determine whether the following two statements are true of false and justify your answer.

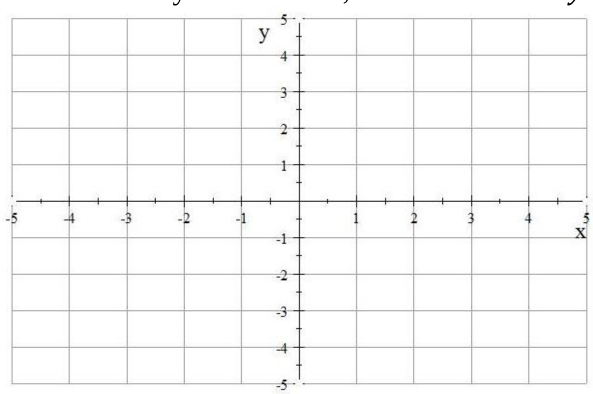
i. T or F: If the *changes* in the measures of two quantities are proportionally related, then the *measures* of the two quantities are proportionally related.

ii. T or F: If the *measures* of two quantities are proportionally related, then the *changes* in the measures of the quantities are also proportionally related.

Problem 3. Suppose that x and y represent the measures of two quantities, and suppose these quantities are linearly related with $\Delta y = -2.50\Delta x$. On the grid provided, sketch the graphs of three different functions

$$y = f(x) \qquad y = g(x) \qquad y = h(x)$$

which satisfy this condition. For each of your functions, is it also true that $y = -2.50x$?

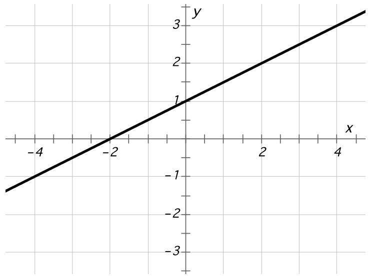


Problem 4. Suppose *x* and *y* represent the measures of two quantities that vary at a constant rate with respect to each other. For Parts (a) – (c) below, use the given information to write a formula that defines the relationship between *x* and *y*.

Part (a). The values of *y* change at a constant rate of –0.9 with respect to *x*, and *y* = 2.4 when *x* = –5.8.

Part (b). We know that *y* = 3.6 when *x* = 12.2, and we also know that *y* = –1.5 when *x* = 8.7.

Part (c). The diagram below shows how the values of y are related to the values of x.



Homework.

Problem 1. Paul was walking in a park. Assume that he walked at a constant speed during the entire trip, and also suppose that during one part of the trip he walked 52.8 feet in 8 seconds.

a. Provide at least four conclusions we can draw from the given information.

b. How far did Paul walk in 14 seconds?

c. Does your answer to Part (b) depend on which 14-second interval we're talking about? Explain.

d. How long did it take Paul to travel any 20-foot distance during his walk?

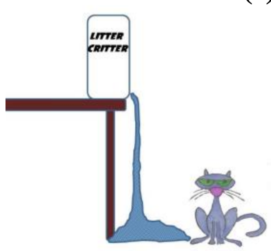
Problem 2. Justo places a bag of cat litter on the kitchen table. Mr. Hipsickles slashes the bottom of the bag, and litter spills onto the floor. Suppose that for every 60 ounces of litter that spills out of the bag, the depth of litter in the bag decreases by 7.8 inches.

a. Properly define the pair of covarying quantities in this process.

b. Construct a formula that represents the relationship between corresponding changes in these quantities. Be careful to identify the variables you use.

c. If the depth of litter in the bag was initially 16 inches, construct a formula that relates the covarying quantities. Is this the same formula you constructed in Part (b)?

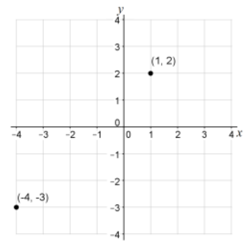
d. There is a varying time quantity that is also associated with this process. Is there an implicit or an explicit relationship between this time quantity and the covarying quantities you identified in Part (a)?



Problem 3. Suppose we know that $\Delta y = m \cdot \Delta x$ for some constant m, and we are given the information in the following table. What is the value of m?

x	y
–3	15.5
1	5.5
3	0.5
8	–12

Problem 4. Suppose we know that $\Delta y = m \cdot \Delta x$ for some constant m, and we are given the information in the following graph. What is the value of m?



Problem 5. Suppose that x and y represent the values of two covarying quantities. Suppose we know that $\Delta y = 4.5 \cdot \Delta x$; and we also know that $y = 4$ when $x = 1$. We want to know the new value of y when $x = -4$. Answer the questions that follow.

$y = 4.5(x - 1) + 4$

$y = 4.5(-4 - 1) + 4$ a. What does $-4 - 1$ represent?

$y = 4.5(-5) + 4$ b. What does $4.5(-5)$ represent?

$y = -22.5 + 4$ c. What does $-22.5 + 4$ represent?

$y = -18.5$ d. What does -18.5 represent?

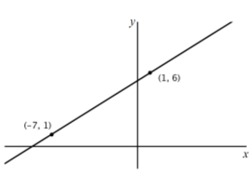
Problem 6. The constant rate of change of y with respect to x is $m = -3.2$, and (–3, –2) is a point on the graph.

a. Write the formula for the relationship between the values of y and the values of x.

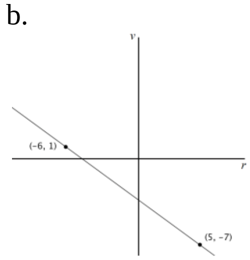
b. Find the value of y when $x = 5$.

Problem 7. Write the formula that defines the linear relationship represented in each of the following graphs.

a.



b.



Answers to the Homework.

Problem 1.

Part (a). Answers will vary, so here are some examples.

(1) Paul's speed remained the same during his entire walk.
(2) During *any* eight-second time interval, Paul walked 52.8 feet.
(3) It would *always* take eight seconds for Paul to walk 52.8 feet.
(4) There quantities "the distance Paul walked, measured in feet since he began his walk" and "the time passed, measured in seconds, since Paul began his walk" covary. (In fact, they covary explicitly, as we show later.)

Part (b). Let D represent the values of the quantity "the distance Paul walked, measured in feet since he began his walk" and let t represent the values of the quantity "the time passed, measured in seconds, since Paul began his walk." The problem tells us that *whenever* $\Delta t = 8$ seconds, we must have $\Delta D = 52.8$ feet.

Consequently, *whatever* fraction of 8 seconds Paul walks, he must walk the *same fraction* of 52.8 feet.

Now, a 14-second time interval is $\frac{14}{8} = 1.75$ times as long as an 8-second time interval. Consequently, Paul must walk 1.75 times are far over any 14-second time interval as he does over any 8-second time interval. Thus, the distance that Paul walks will be $(1.75)(52.8) = 92.4$ feet.

Part (c). No, it does not matter which time interval we consider.

Part (d). Whatever fraction of any 52.8 foot distance Paul walks will take *the same fraction* of an 8-second time interval to accomplish. Now, a change of 20 feet in distance is $\frac{20}{52.8} \approx 0.379$ as far as a change of 52.8 feet in distance. Therefore, the time it takes to walk this distance will be approximately $(0.379)(8) = 3.032$ seconds.

Problem 2.

Part (a). Let w represent the values of the quantity "the weight in ounces of cat litter in the bag" and let d represent the values of the quantity "the depth of litter in the bag, measured in inches."

Part (b). We know that *whenever* $\Delta w = 60$ ounces, we know that $\Delta d = 7.8$ inches. Therefore, we also know that

$$60\Delta d = 7.8\Delta w \quad \text{OR} \quad \Delta d = \left(\frac{7.8}{60}\right)\Delta w$$

Part (c). Since the initial value of d is provided, it is easiest to express the values of d as a function of the values of w. The formula that relates the values of d in terms of the values of w would be

$$d = f(w) = 16 + \left(\frac{7.8}{60}\right)w$$

This formula is clearly not the same as the formula relating the changes in the quantities that you constructed in Part (b).

Part (d). The covariance is implicit, because we do not have enough information to write either values of the weight or the valuesof the depth quantity as a function of the values of the time quantity.

Problem 3. Since we know there is a constant relationship between the changes in the values of y and the changes in the values of x, we may compute *any* pair of these changes to determine the value of m. For example,

- As the values of x change from $x = -3$ to $x = 3$, we know that $\Delta x = 6$ and we know that $\Delta y = -15$. Therefore,

$$m = -\frac{15}{6} = -2.5$$

Problem 4. Based on the information given, we know

$$m = \frac{\Delta y}{\Delta x} = \frac{2 - (-3)}{1 - (-4)} = 1$$

Problem 5.

Part (a). –4 – 1 represents the change in x from $x = 1$ to $x = -4$.
Part (b). 4.5(–5) represents the change in y from $y = 4$ to the new value of y.
Part (c). –22.5 + 4 represents the value of y when $x = -4$ (because it shows the change in y added to the "initial" value of y).
Part (d). –18.5 represents the value of y when $x = -4$.

Problem 6.

Part (a). There are many ways to approach this problem. The easiest way is via the *point-slope* formula which tells us that

$$y — 2 = -3.2(x — 3) \quad \text{OR} \quad y = f(x) = -3.2x + 11.6$$

Part (b). When $x = 5$, we know that $y = f(5) = -4.4$.

Problem 7. Once again, the point-slope formula is the easiest way to construct the relationships.

Part (a). We know

$$m = \frac{\Delta y}{\Delta x} = \frac{6 - 1}{1 - (-7)} = \frac{5}{8}$$

$$y - 1 = \frac{5}{8}\left(x - (-7)\right) \quad \text{OR} \quad y = \frac{5}{8}x + \frac{43}{8}$$

Part (b). We know

$$m = \frac{\Delta v}{\Delta r} = \frac{1 - (-7)}{-6 - 5} = -\frac{8}{11}$$

$$v - 1 = -\frac{8}{11}\left(r - (-6)\right) \quad \text{OR} \quad v = -\frac{8}{11}r - \frac{37}{11}$$