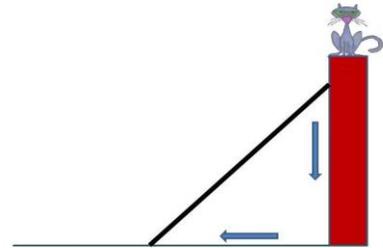


**Problem 1.** Justo has a twelve-foot ladder leaning against a vertical wall as shown below. Mr. Hipsickles scampers up the ladder, causing the base of the ladder to slide away from the wall.



**Part (a).** Let  $x$  represent the measures of the quantity “the horizontal distance in feet from the base of the ladder to the wall” and let  $y$  represent the measures of the quantity “the vertical distance in feet from the top of the ladder to the ground.” Is there a linear relationship between these quantities? Justify your answer.

**Part (b).** If you did not do so in Part (a), construct a function  $y = f(x)$  that gives the values of  $y$  in terms of the values of  $x$ .

**Part (c).** Use a graphing calculator to sketch the graph of your function  $f$  in the viewing window defined by  $0 \leq x \leq 12$  and  $0 \leq y \leq 12$ . What is the meaning of the point  $(10, f(10))$ ?

**Part (d).** Use the zoom feature on the graphing calculator to zoom in on the graph of the function  $f$ , keeping the point  $(10, f(10))$  at the center of the zoom. If you zoom in several times, what do you notice about the graph of the function  $f$ ?

When examining function behavior, it is often very useful to assume that the measures of the output quantity vary at a constant rate with respect to the measures of the input quantity --- at least over small intervals of the domain --- even when we know it does not.

#### **Local Constant Rate of Change**

Let  $y = f(x)$  be a function. We assume a *local constant rate of change* for the function  $f$  when we suppose that the relationship between the measures of the output quantity and the measures of the input quantity is approximately linear on some part of the domain.

In other words, we suppose there exists a linear function  $y = T(x)$  so that  $f(x) \approx T(x)$  for all values of  $x$  in that part of the domain. The constant rate of change for the function  $T$  is what we call the local constant rate of change for the function  $f$ .

We use the phrase “local constant rate of change” because we do *not* assume that the measures of the output quantity vary at a constant rate with respect to the measures of the input quantity over the entire domain, but only over a part of the domain.

It is useful to assume a local constant rate of change is because then the changes in the quantity's measures are proportional; and this makes it easy to reason about the effects of the covariance... at least over small input intervals.

**Problem 2.** The function  $f$  from Problem 2 exhibits a local constant rate of change near the point  $(10, f(10))$ . What is the approximate value of this local constant rate of change?

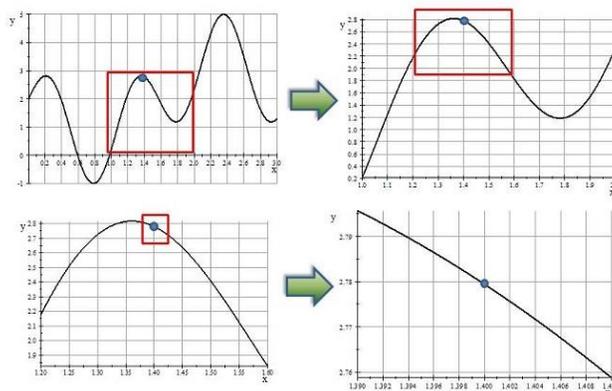
**Problem 3.** Construct the formula for the linear function  $y = T(x)$  that has the constant rate of change you found in Problem 3 and passes through the point  $(10, f(10))$ .

**Problem 4.** Keeping the same zoom window you had in Problem 2, use the graphing calculator to sketch the graph of the function  $T$  along with the function  $f$ . What do you notice? Return to the original viewing window. What do you notice about the graphs of  $f$  and  $T$  now?

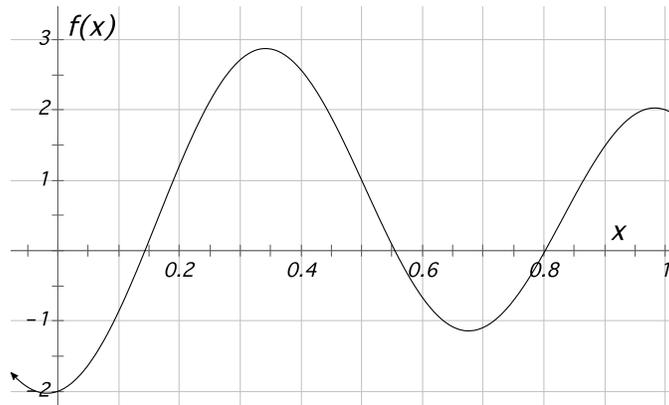
### Local Linearity

Suppose that  $y = f(x)$  is function. We say that the function  $f$  is *locally linear* at an input value  $x = a$  provided the graph of  $f$  looks more and more like a straight line the closer we zoom in on the point  $(a, f(a))$ . This straight line is called the *tangent line* to the graph of  $f$  at  $x = a$ . The specified point  $(a, f(a))$  is called the *point of tangency*.

Here is an example of a function that is locally linear at the input value  $x = 1.4$ . As we zoom in closer and closer on the point  $(1.4, 2.779)$ , the graph of the function starts to look like a straight line.



**Problem 5.** The following is a graph of  $y = f(x) = \sin(\pi x) - 2 \cos(3\pi x)$  on the input interval  $[0, 1]$ .



**Part (a).** Enter the formula for the function  $f$  into your graphing calculator. (Make sure your calculator is in radian mode.) Use the zoom feature on the calculator to zoom in on the point  $(0.4, f(0.4))$  until the graph of  $f$  looks like a straight line.

**Part (b).** Use your graph from Part (a) to estimate the local constant rate of change of  $f(x)$  with respect to  $x$  near  $x = 0.4$ .

**Part (c).** Construct a formula for the tangent line to the graph of  $f$  at the point  $(0.4, f(0.4))$ .

**Part (d).** Starting with your final zoom-in, graph the function  $f$  along with the tangent line to the graph of  $f$  you constructed in Part (c). Now, return to the original viewing window. Carefully draw what you see on the graph provided above.

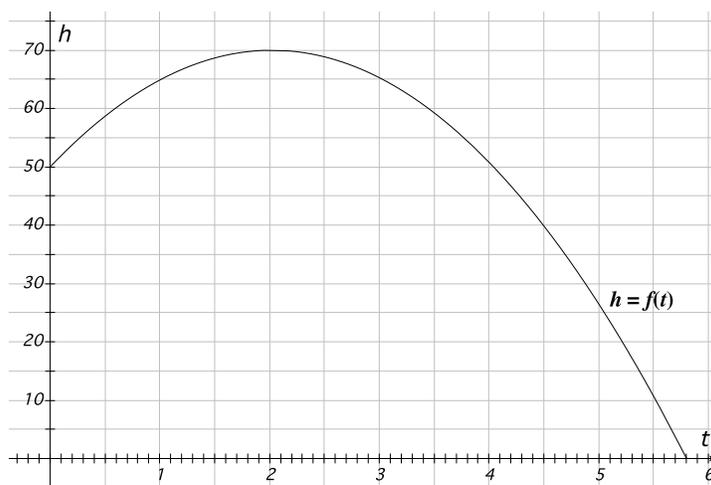
**Part (e).** Do you think it is possible to keep zooming in forever on the graph of  $f$  at the point  $(0.4, f(0.4))$ ? What problems might you encounter while attempting to do so? Are these problems with your graphing device or are they problems with the function itself? Explain.

**Part (f).** Carefully draw the tangent line to the graph of the function  $f$  at the point  $(0.2, f(0.2))$ , and use your graph to estimate the local constant rate of change of  $f$  with respect to  $x$  near  $x = 0.2$ . How did you obtain your estimate?

### Homework.

Problems 1 – 4 refer to the following scenario.

Toby is standing on top of a five-story building. He leans over the edge of the building and tosses a penny vertically into the air, then steps back and watches it rise and then fall to the ground. The following graph of the function  $f$  shows the relationship between the values of the height  $h$  in feet of the penny above the ground and the values of time  $t$  in seconds since Toby tossed the penny.



**Problem 1.** Does the graph above show the *actual* path that the penny took? Explain your answer.

**Problem 2.** Use the table below to determine the average velocity of the penny on the given time intervals. Show your calculations.

$t$	4.00	3.40	3.20	3.10	3.00
$f(t)$	50.80	60.68	63.18	64.29	65.30

**Part (a).** Time interval  $3.00 \leq t \leq 4.00$ .

**Part (b).** Time interval  $3.00 \leq t \leq 3.40$ .

**Part (c).** Time interval  $3.00 \leq t \leq 3.20$ .

**Part (d).** Time interval  $3.00 \leq t \leq 3.10$ .

**Problem 3.** Use the table below to determine the average velocity of the penny on the given time intervals. Show your calculations.

$t$	2.00	2.40	2.80	2.90	3.00
$f(t)$	70.00	69.30	67.02	66.21	65.30

**Part (a).** Time interval  $2.00 \leq t \leq 3.00$ .

**Part (b).** Time interval  $2.40 \leq t \leq 3.00$ .

**Part (c).** Time interval  $2.80 \leq t \leq 3.00$ .

**Part (d).** Time interval  $2.90 \leq t \leq 3.00$ .

**Problem 4.** Carefully draw the tangent line to the graph of the function  $f$  at the point  $(3, 65.30)$ .

**Part (a).** Use the tangent line to approximate the local constant rate of change of  $f(t)$  with respect to  $t$  around  $t = 3$  seconds.

**Part (b).** How do the average velocities you computed in Parts (b) and (c) compare to the local constant rate of change you approximated in Part (d)?

**Problem 5.** Consider the function  $a = f(b) = b^2$ .

**Part (a).** What is the meaning of the function  $A$  defined below?

$$y = A(x) = \frac{f(4+x) - f(4)}{(4+x) - 4}.$$

**Part (b).** Which of the following formulas is algebraically equivalent to the formula that defines the function  $A$  in Part (a)?

i.  $y = \frac{16 + x^2 - 16}{x}$

ii.  $y = \frac{16 + 8x + x^2 - 16}{x}$

iii.  $y = \frac{x^2 + 8x}{x}$

**Part (c).** Explain how you could use the table feature on your graphing calculator and one of the equivalent formulas for the function  $A$  to estimate the local constant rate of change of  $f(a)$  with respect to  $a$  around  $a = 4$ .

**Part (d).** Approximate the local constant rate of change of  $f(a)$  with respect to  $n$  around  $n = 4$ . How accurate do you think your approximation is? Explain your thinking.

**Answers to the Homework.**

**Problem 1.** No, the graph shown represents the covariation between the vertical height of the penny (measured in feet from the ground) and the number of seconds passed since the penny was tossed.

**Problem 2.**

Time interval  $3.00 \leq t \leq 4.00$ .

The average rate of change of the penny on this interval is

$$\frac{Df(t)}{Dt} = \frac{50.80 - 65.30}{4 - 3} = -14.5.$$

Time interval  $3.00 \leq t \leq 3.40$ .

The average rate of change of the penny on this interval is

$$\frac{Df(t)}{Dt} = \frac{60.68 - 65.30}{3.4 - 3} = -11.55.$$

Time interval  $3.00 \leq t \leq 3.20$ .

The average rate of change of the penny on this interval is

$$\frac{Df(t)}{Dt} = \frac{63.18 - 65.30}{3.2 - 3} = -10.6.$$

Time interval  $3.00 \leq t \leq 3.10$ .

The average rate of change of the penny on this interval is

$$\frac{Df(t)}{Dt} = \frac{64.29 - 65.30}{3.1 - 3} = -10.10.$$

**Problem 3.**

Time interval  $2.00 \leq t \leq 3.00$ .

The average rate of change of the penny on this interval is

$$\frac{Df(t)}{Dt} = \frac{65.30 - 70.00}{3 - 2} = -4.7.$$

Time interval  $2.40 \leq t \leq 3.00$ .

The average rate of change of the penny on this interval is

$$\frac{Df(t)}{Dt} = \frac{65.30 - 69.30}{3 - 2.4} = -6.67.$$

Time interval  $2.80 \leq t \leq 3.00$ .

The average rate of change of the penny on this interval is

$$\frac{Df(t)}{Dt} = \frac{65.30 - 67.02}{3 - 2.8} = -8.6.$$

Time interval  $2.90 \leq t \leq 3.00$ .

The average rate of change of the penny on this interval is

$$\frac{Df(t)}{Dt} = \frac{65.30 - 66.21}{3 - 2.9} = -9.1.$$

**Problem 4.** The tangent line should pass close to the point (4,55.0); hence, the slope of the tangent line to the graph of the function  $f$  at the point (3,65.3) will be approximately  $m = -10.3$  feet per second. As the time interval over which the average rates of change are taken get smaller, the values of the average rate of change become better and better approximations to the slope of the tangent line.

**Problem 5.**

**Part (a).** The function  $A$  gives the average rate of change for the function  $f$  on the input interval from  $b = 4$  to  $b = 4 + x$ .

**Part (b).** Only Formulas (ii) and (iii) are equivalent to the original formula.

**Part (c).** You would begin by setting one of the calculator functions (say Y1) equal to one of the equivalent formulas. Next, you would create the average rate of change function as a second formula in the calculator. For example, you could let

$$Y2 = (Y1(4+x) - Y1(4)) / x$$

Next, you would set the table start to a value close to  $x = 0$ , say  $x = -0.01$ . You would then set the variable increment to a value one-tenth as large as  $|x|$ . (In this case, you would let  $\Delta x = 0.001$ .) Finally, you would look at the average value output from the table to determine whether the average values are approaching some fixed number. You may have to initiate the process with values of  $b$  much closer to 0 in order to see a definitive pattern.

**Part (d).** If we initiate the table with  $x = -0.01$  and  $\Delta x = 0.001$  as suggested in Part (c) above, we would obtain the following table output.

X	-0.004	-0.003	-0.002	-0.001	0.000	0.001	0.002	0.003	0.004
Y2	7.996	7.997	7.998	7.999	ERROR	8.001	8.002	8.003	8.004

Based on this table, it seems reasonable to assume that the local constant rate of change would be approximately 8. Averaging values of  $Y2(-0.001)$  and  $Y2(0.001)$  gives us the approximation

$$m = 8$$

It is difficult to say from this table just how accurate this estimate would be; however the average value outputs appear to be stable to two decimal places. Therefore, our estimate is probably also accurate to at least two decimal places.