Let $y = f(x)$ be a function, and suppose that $x = a$ is an input value in the domain of the function f. Recall that the difference-quotient function

$$g_a(h) = \frac{f(a+h) - f(a)}{h}$$

gives the average rate of change for the function f with respect to the values of x on the input interval from $x = a$ to $x = a + h$.

Let's think about the average value function $y = g_a(h)$ for the function $= f(x) = \frac{1}{2x}$. For any fixed input value $x = a$, we know that

$$g_a(h) = \frac{f(a+h) - f(a)}{h} = \frac{1}{h}\left[\frac{1}{2(a+h)} - \frac{1}{2a}\right]$$

Problem 1. Use algebra to write the formula above as a single fraction. In particular, show that, as long as $h \neq 0$, we have

$$\frac{1}{h}\left[\frac{1}{2(a+h)} - \frac{1}{2a}\right] = -\frac{1}{2a(a+h)}$$

Now, suppose we let

$$G_a(h) = -\frac{1}{2a(a+h)}$$

We know the output of the function g_a is the same as the output of the function G_a for every input value of h *except* for $h = 0$. The function g_a has no output when $h = 0$, but the function G_a *does* have output when $h = 0$. In fact, we know

$$G_a(0) = -\frac{1}{2a^2}$$

Let's think about what this means for a moment. To help us do this, let's get specific about the fixed value of x so we can compare outputs for the two functions. Let $x = 3$. (You can choose any fixed value for x that you like.) We are now considering the two functions

$$g_3(h) = \frac{1}{h}\left[\frac{1}{2(3+h)} - \frac{1}{6}\right] \qquad G_3(h) = -\frac{1}{6(3+h)}$$

We know that the function g_3 is undefined when $h = 0$, and we know that

$$G_3(0) = -\frac{1}{18} \approx -0.05556$$

Problem 2. While it is true that the function g_3 is undefined when $h = 0$, it is also true that this function *is* defined for values of h that are very close to 0.

Part (a). Fill in the table of output values for the function g_3 corresponding to input values of h that are very close to 0.

Value of h	-0.015	-0.010	-0.005	-0.001	0	0.001	0.005	0.010	0.015
Value of $g_3(h)$									

Part (b). As the values of h get closer and closer to 0, how do the values of $g_3(h)$ compare to the values of $G_3(0)$?

Even though the function g_3 is not defined when $h = 0$, the *process* of choosing values of h closer and closer to 0 tells us that the output of the function g_3 gets closer and closer to the value $G_3(0)$. In mathematics, we use a special notation to indicate this.

$$\lim_{h \to 0} g_3(h) = \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{2(3+h)} - \frac{1}{6}\right] = -\frac{1}{18}$$

We read this notation as "*the limiting process for the values of $g_3(h)$ as the values of h approach 0 produces a value of* $-1/18$."

It is customary to shorten this reading to "*the limit of the function g_3 as h approaches 0 is equal to* $-1/18$."

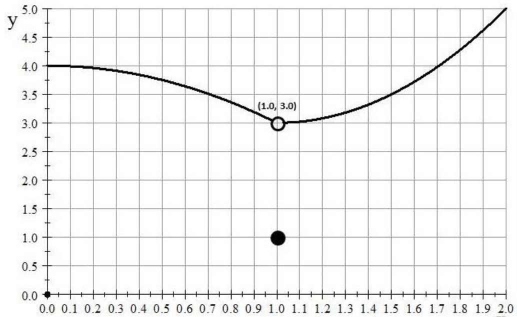
Limiting Process

Suppose that $y = f(x)$ is a function, and suppose that L is a real number. When we write

$$\lim_{x \to a} f(x) = L$$

we mean that the values of $f(x)$ get closer and closer to L as the values of x get closer and closer to a.

Consider the graph of the function $y = f(x)$ shown in the diagram below.



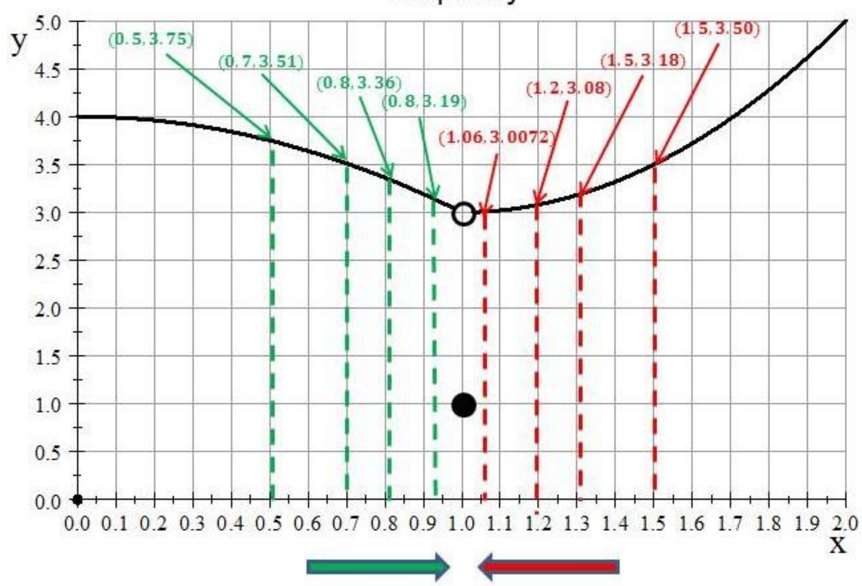
Graph of f

Based on this diagram, what can we say about the process

$$\lim_{x \to 1} f(x)$$

To answer this question, we have to think about what is happening to the values of $f(x)$ as the values of the input variable x get closer and closer to $x = 1$.

Graph of f



Values of x approach 1.0

Even though the diagram shows us that $f(1.0) = 1$, the limiting process produces a different value. In particular,

$$\lim_{x \to 1} f(x) = 3.0$$

The value of a limiting process does not depend on the output value of the function at the limiting input value. Indeed, in the opening example, we know that $g_3(0)$ does not exist, yet

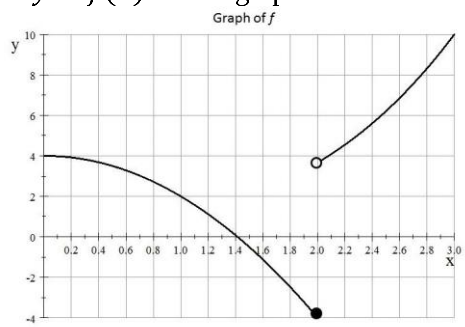
$$\lim_{h \to 0} g_3(h) = -\frac{1}{18}$$

Furthermore, in the last example, we know $f(1.0) = 1$, but we also know that

$$\lim_{x \to 1} f(x) = 3.0$$

The value of a limiting process depends only on the *behavior* of the output values for the function as the input values get closer and closer to the limiting input value.

Problem 3. Consider the function $y = f(x)$ whose graph is shown below.



Part (a). What is the value of $f(1)$?

Part (b). Based on the graph, what is the value of $\lim_{x \to 1} f(x)$?

Part (c). What is the value of $f(2)$?

Part (d). Suppose we let

$$\lim_{x \to 2^-} f(x)$$

represent the process of considering the output values $f(x)$ *only as the values of* x *approach* $x = 0$ *from the left.* What is the value of this limiting process?

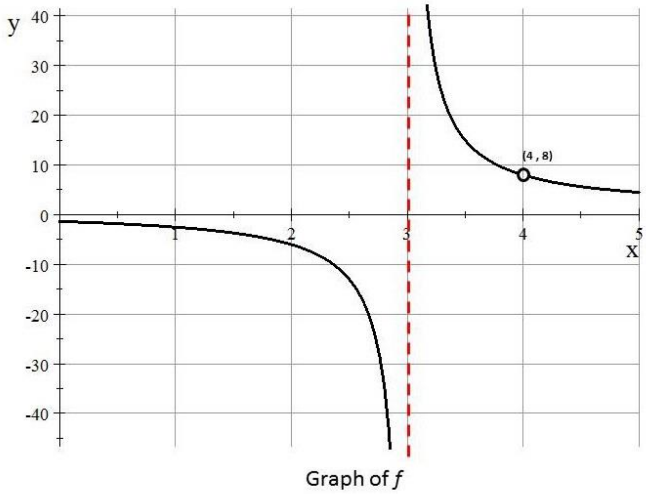
Part (e). Suppose we let

$$\lim_{x \to 2^+} f(x)$$

represent the process of considering the output values $f(x)$ *only as the values of* x *approach* $x = 0$ *from the right.* What is the value of this limiting process?

Part (f). Is there a value for the limiting process $\lim_{x \to 2} f(x)$? Explain your thinking.

Problem 4. Now, consider the function $y = f(x)$ whose graph is shown in the diagram below.



Part (a). Do you think there is a value for the limiting process $\lim_{x\to 4} f(x)$? Explain your thinking.

Part (b). Do you think there is a value for the limiting process $\lim_{x\to 3^+} f(x)$? Explain your thinking.

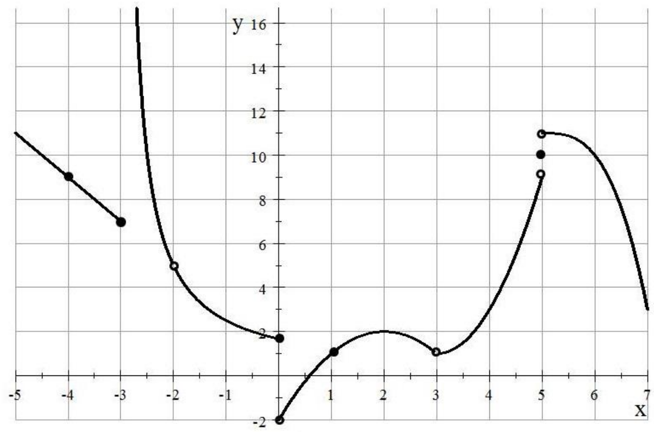
Part (c). Do you think there is a value for the limiting process $\lim_{x\to 3^-} f(x)$? Explain your thinking.

It is important to realize that this limiting process does not produce a real number --- the limit does not exist. We often use the symbol "$\pm\infty$" to indicate that the output values of f get more and more positive, or more and more negative. For example, we could write

$$\lim_{x\to 3^-} f(x) = -\infty \qquad\qquad \lim_{x\to 3^+} f(x) = +\infty$$

to describe the behavior we see from the graph of the function f in Problem 4 above.

Problem 5. Consider the graph of the function $y = f(x)$ shown in the diagram below.



Graph of f

Based on this graph, determine the value of the following limiting processes, if these values exist.

(a) $\lim_{x\to -4} f(x)$ (b) $\lim_{x\to -2} f(x)$ (c) $\lim_{x\to 5^+} f(x)$ (d) $\lim_{x\to 0^-} f(x)$

(e) $\lim_{x\to -3^-} f(x)$ (f) $\lim_{x\to 0} f(x)$ (g) $\lim_{x\to -3^+} f(x)$ (h) $\lim_{x\to 3} f(x)$

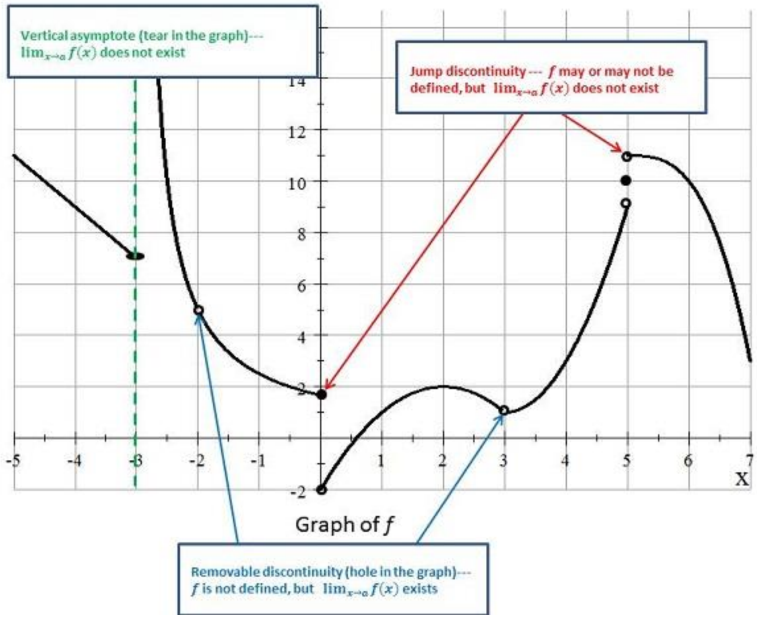
Continuity at an Input Value

Suppose that $y = f(x)$ is a function. We say that the function f is *continuous* at the input value $x = a$ provided the following conditions are met:

1. The function f is defined at $x = a$; that is, the output value $f(a)$ exists.
2. The limiting process as the values of x approach $x = a$ produces the value $f(a)$; that is,

$$\lim_{x\to a} f(x) = f(a)$$

If the function f is not continuous at an input value $x = a$, then we say that the function f has a *discontinuity* at the value $x = a$. There are three types of discontinuity that we commonly encounter, and they are described in the diagram below.



- *Jump discontinuity* --- When the function f has a jump discontinuity at the input value $x = a$, then

$$\lim_{x \to a} f(x)$$

  does not exist, but the limits from the left or right usually do exist. The function may or may not be defined at $x = a$.

- *Vertical asymptote* --- When the function f has a vertical asymptote at the input value $x = a$, then there is a "tear" in the graph at this input value. The function f may or may not be defined at the input value $x = a$, but it will be the case that at least one of the following is true:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad OR \quad \lim_{x \to a^+} f(x) = \pm\infty$$

- *Removable discontinuity* --- When the function f has a removable discontinuity at the input value $x = a$, then there is a "hole" in the graph at this input value. The function f may or may not be defined at the input value $x = a$, but it will be the case that

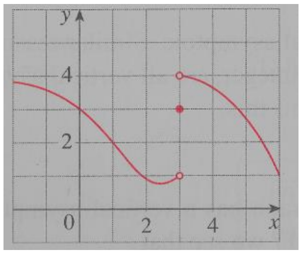
$$\lim_{x \to a} f(x)$$

  exists. If $f(a)$ happens to exist, it will also be the case that

$$\lim_{x \to a} f(x) \neq f(a)$$

Homework.

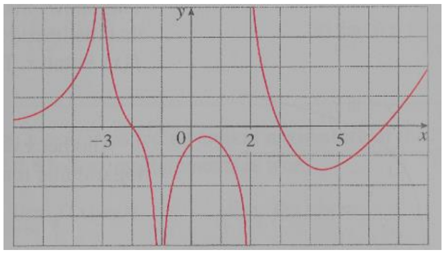
Problem 1. Consider the graph of the function $y = f(x)$ shown below.[1]



Use the graph to determine the values of the following limiting processes.

(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 3^-} f(x)$ (c) $\lim_{x\to 3^+} f(x)$ (d) $\lim_{x\to 3} f(x)$

Problem 2. Consider the graph of the function $y = f(x)$ shown below.



Use the graph to determine the values of the following limiting processes.

(a) $\lim_{x\to 3} f(x)$ (b) $\lim_{x\to -3^-} f(x)$ (c) $\lim_{x\to -1^+} f(x)$ (d) $\lim_{x\to 2^-} f(x)$

Problem 3. The function f whose graph is shown in Problem 1 has a discontinuity at the input value $x = 3$. Is this a removable discontinuity? Justify your answer.

Problem 4. Consider the function f whose graph is shown in Problem 2.

Part (a). Does the function f have any removable discontinuities? If so, at what input values do these discontinuities occur?

Part (b). Does the function f have any jump discontinuities? If so, at what input values do these discontinuities occur?

Part (c). Does the function f have any vertical asymptotes? If so, at what input values do these discontinuities occur?

[1] Images taken from *Calculus Early Transendentals* 8th Ed., Stewart, Page 92.

Answers to the Homework.

Problem 1.

(a) $\lim_{x \to 0} f(x) = 3$ (b) $\lim_{x \to 3^-} f(x) = 1$ (c) $\lim_{x \to 3^+} f(x) = 4$ (d) $\lim_{x \to 3} f(x)$ Does not exist

Problem 2.

(a) $\lim_{x \to 3} f(x) = 0$ (b) $\lim_{x \to -3^-} f(x) = +\infty$ (c) $\lim_{x \to -1^+} f(x) = -\infty$ (d) $\lim_{x \to 2^-} f(x) = -\infty$

Problem 3. The discontinuity displayed by the function is a jump discontinuity. Since the two-sided limit does not exist, this discontinuity cannot be removed.

Problem 4. The only discontinuities displayed by this function are vertical asymptotes. The vertical asymptotes occur at $x = -3$, $x = -1$, and $x = 2$.