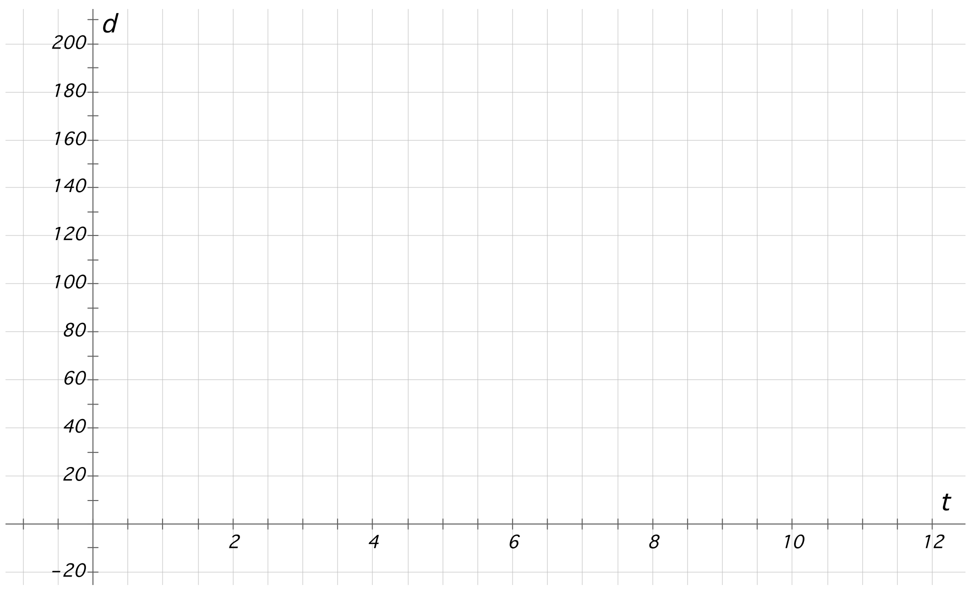
In this investigation, we will combine the concept of local linearity and the limiting process to introduce the key idea for this course.

Problem 1. A car is driving away from a traffic light. The distance *d* (in feet) of the car from the traffic light *t* seconds since the car started moving is given by the formula

$$d = f(t) = 1.3t^2 - 17$$

Part (a). Draw a graph of the relationship between the car's distance *d* (in feet) from the traffic light and the time *t* (in seconds) since the car started moving.



Part (b). Approximate the car's speed *exactly* 8 seconds after it started moving and explain the strategy you used. Also, illustrate what your approximation represents on the graph you drew in Part (a) and explain how what you drew corresponds to the approximation value you computed.

> ***Instantaneous Rate of Change***
>
> Let $y = f(x)$ be a function, and let $x = a$ be a fixed value in the domain of the function f. The *instantaneous* rate of change of the output values $f(x)$ with respect to x at the value $x = a$ is defined by
>
> $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Consider the context in Problem 1. The only way we could determine the car's speed 8 *exactly* seconds after it started moving was to approximate it by computing an average rate of change over a very small interval of time around the input value *t* = 8 seconds.

Without a small change in time, there is no corresponding change in distance, and thus no rate of change. Since rates of change always occur over an interval of the input variable (even really small intervals), you should interpret "instantaneous rate of change" as "average rate of change over an interval so small that the changes in the quantities' measures are essentially proportional."

Notice that the instantaneous rate of change in the output of f at $x = a$ with respect to the input values is a *derived quantity* --- it is obtained as the end result of a limiting process on average rates of change. For this reason, this quantity is often called the ***derivative of f with respect to x at x = a*** .

There are two commonly used notations to represent the derivative of the function f with respect to x at $x = a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \left. \frac{df}{dx} \right|_{x=a}$$

Notice that the second notation resembles the ratio $\frac{\Delta f(x)}{\Delta x}$.

This resemblance is deliberate because the instantaneous rate of change for the outputs $f(x)$ with respect to x *at* the input value $x = a$ **is the same as** the local constant rate of change for the outputs $f(x)$ with respect to x *near* the input value $x = a$.

In particular, when we consider small changes in the input variable for values of x very close to $x = a$, we have

$$\Delta f(x) \approx f'(a)\Delta x$$

The symbols "dx" and "df" still refer to changes in the measures of the input and output quantities, but we use *d* instead of Δ to emphasize these changes are so small that they are essentially proportionally related. It is traditional to say that dx and df represent *differential* changes in the values of the quantities.

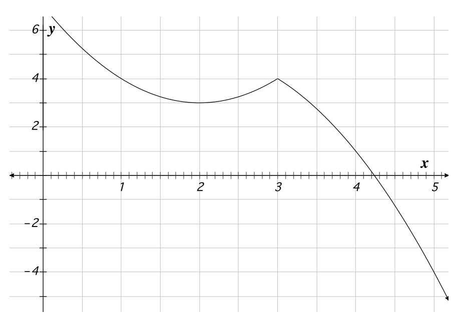
We have two different notations to represent values of the same derived quantity because calculus was simultaneously developed by two individuals, Isaac Newton and Gottfried Leibniz, who represented their ideas differently. Newton favored a version of the "prime" notation, while Leibniz favored the "differential" notation.

Problem 2. Consider the function $f(x) = 1 + |x - 1|$.

Part (a). Enter this function into your graphing calculator and use your calculator to sketch its graph in the viewing window $-1 \le x \le 2$, $0 \le y \le 3$. Do you think that the function f is locally linear at the point $(0, 2)$? Explain.

Part (b). Is the function f locally linear at the point $(1, 1)$? Explain.

Problem 3. Consider the graph of the function $y = f(x)$ shown below.



Part (a). Estimate the values of $f'(1)$ and $\left.\frac{df}{dx}\right|_{x=2}$ and explain how you determined your estimates.

Part (b). Is it possible to estimate the value of $\left.\frac{df}{dx}\right|_{x=3}$? If so, estimate this value. If not, explain why.

Homework.

Problem 1. Justo is keeping track of the weight of his cat, Mr. Hipsickles. Let m represent the number of months since January 1, 2015, and let let $f(m)$ represent Mr. Hipsickles' weight (in pounds) m months after January 1, 2015. Explain the meaning of $f'(15)$. Do not use the word "instantaneous" in your explanation.

Problem 2. Imagine a bottle filling with water. Let x represent the height of the water in the bottle (in centimeters) and let $g(x)$ represent the volume of water in the bottle (in milliliters). Explain the meaning of $\left.\frac{dg}{dx}\right|_{x=3.7}$. Do not use the word "instantaneous" in your explanation.

Problem 3. Use your graphing calculator to sketch a graph of the function

$$f(x)=\begin{cases} 1+x^2, & x\le 1 \\ 3x-1, & x>1 \end{cases}$$

Is f locally linear at the input value $x = 1$? Explain your reasoning. (You may need to look up methods for graphing piecewise-defined functions on your calculator.)

Problem 4. Is the function $f(x)=\sqrt[3]{x}$ locally linear at the input value $x = 0$? Explain your reasoning.

Problem 5. Let $g(x)=\ln(x)$.

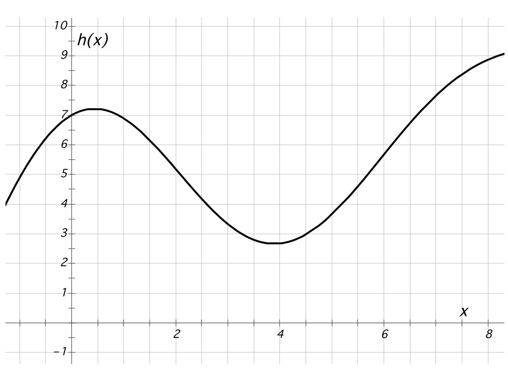
a. Use the zoom feature on your graphing calculator to estimate the value of $g'(1)$.

b. Construct the formula for the linear function $y = T(x)$ that is tangent to the graph of g at the point (1, 0).

c. Using your current zoom window, graph the tangent line along with the function g. What do you notice?

d. Change the viewing window on your calculator to $0 \le x \le 4$ and $-5 \le y \le 2$. What do you notice?

Problem 6. The following is a graph of the function h. Represent on this graph the value $h'(4)$ and explain how what you drew represents $h'(4)$.



Answers to the Homework.

Problem 1. Interpreting a derivative in practical terms always involves understanding how small changes in the values of the input variable affect the output of the function. We use the notion of local linearity to accomplish this. The fact that $f'(15)$ exists tells us that the function f is locally linear near the point $(15, f(15))$. This means that, for a small change Δm in the number of months since January 1, 2015, we have $f(15+\Delta m) \approx f(15) + f'(15)\Delta m$.

Problem 2. See Problem 1 above.

Problem 3. The graph of this function is misleading if your viewing window is too large. The only input value where the function f could fail to be locally linear is $x = 1$. If you zoom in sufficiently close to this point, you will discover that there is a "kink" in the graph at the point $(1, f(1))$. (It typically takes three or four zooms from a standard viewing window to see the kink.)

Problem 4. If you sketch the graph of this function on your calculator and proceed to zoom in on the point $(0, f(0))$, you will find that the graph of the function f does indeed start to look like a straight line near this point. However, the straight line is *vertical*. Since vertical lines have no definable slope, we consider the function f NOT to be locally linear at the input value $x = 0$.

It is debatable whether we ought to modify the definition of local linearity to clearly exclude "vertical" tangent lines. Some mathematicians phrase the definition to read "A function f is *locally linear* at an input value $x = a$ provided the graph of f looks more and more like a *nonvertical* straight line the closer we zoom in on the point $(a, f(a))$." Other mathematicians see no need to do this since vertical lines have no definable slope.

Problem 5. The formula for the tangent line to the graph of the function $g(x) = \ln(x)$ at the point $(1, \ln(1))$ is $y = T(x) = x - 1$. When zoomed in very close to the point of tangency, the output of the function g and the output of the function T will be almost equal. In a viewing window that encompasses a larger input interval, the output values from the two graphs will become increasingly different as the input values increase or decrease from $x = 1$.

Problem 6. The slope of the line tangent to the graph of the function *h* when *x* = 4 represents the value *h*´(4). Because *h* is locally linear near *x* = 4, the local constant rate of change of *h*(*x*) with respect to *x* near
x = 4 is essentially the same as the constant rate of change represented by the slope of the line tangent to the graph of *h* that passes through the point (4, 2.68).