

We say that a function $y = f(x)$ is locally linear at a point $(a, f(a))$ provided “zooming in” on the graph of f at the point $(a, f(a))$ produces a view of the graph that looks more and more like a straight line. When a function is locally linear, we know several things:

1. Near $x = a$, small changes in the value of the input x will be approximately proportional to corresponding changes in the output value of f . In other words, for values of x close to $x = a$, we have

$$\Delta f(x) \approx m\Delta x$$

for some constant m . This constant m is called the local constant rate of change for $f(x)$ with respect to x near $x = a$.

2. It is possible to construct a tangent line to the graph of f at the point $(a, f(a))$. The point-slope formula for this line is

$$y = m(x - a) + f(a)$$

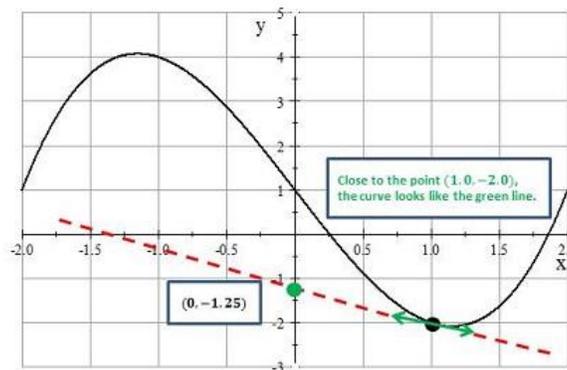
where m is the local constant rate of change for $f(x)$ with respect to x near $x = a$. For values of x close to $x = a$, the output from the tangent line is a good approximation to the output of f .

3. The instantaneous rate of change of the output values $f(x)$ with respect to x for the function f exists at the input value $x = a$. This number, denoted by $f'(a)$, is the limiting value of the average rates of change for the function f over increasingly small input intervals that begin or end at $x = a$. In symbols, we have

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

4. The instantaneous rate of change of the output values $f(x)$ with respect to x for the function f at $x = a$ is the same as the local constant rate of change for $f(x)$ with respect to x near $x = a$.

Consider, the graph of a function $y = f(x)$ shown below. Suppose we want to estimate the value of $f'(1.0)$.



Based on the sketch of the tangent line drawn in the diagram above, we can see that

$$f'(1.0) \approx \frac{-2.00 - (-1.25)}{1 - 0} = -0.75$$

Problem 1. On the graph provided above, sketch the line tangent to the graph of the function f at the point $(-0.5, 2.90)$ and use your sketch to determine the approximate value of $f'(-0.5)$.

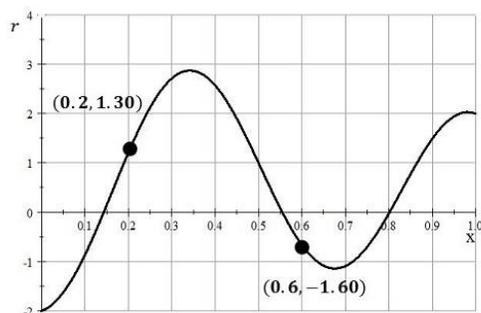
The Derivative Function for a Function

Let $y = f(x)$ be a function. The function $r = f'(x)$ is defined to be the function whose output for any input value $x = a$ is the instantaneous rate of change for f with respect to x at $x = a$. This function is called the *derivative function* for the function f . It is also common to use the symbol

$$r = \frac{df}{dx} = \frac{d}{dx}[f(x)]$$

to represent the derivative function.

Problem 2. The diagram below shows the graph of the derivative function $r = f'(x)$ for a function $y = f(x)$. There are at least three ways we can interpret the point $(0.2, f'(0.2))$. What are they?



Problem 3. Suppose we know that $f(0.6) = 1.43$. Use the graph of the derivative function for the function f provided in Problem 2 to help you construct the point-slope formula for the line tangent to the graph of the function f at the point $(0.6, f(0.6))$.

Example 1. Consider the function $y = f(x) = x^3 - 4x + 1$. Sketch the graph of the derivative function $r = f'(x)$.

Solution. One way to accomplish this task would be to estimate output values for the derivative function f' for some input values of x , plot the ordered pairs $(x, f'(x))$ and see if we can detect a pattern. Since

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

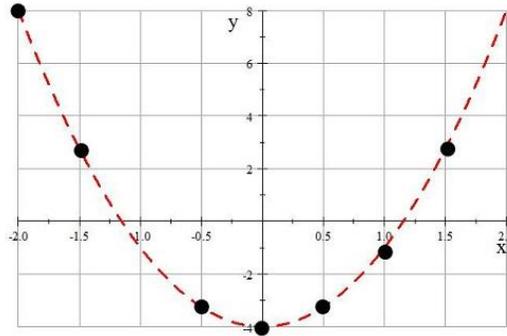
we can compute very good estimates for the values of $f'(a)$ by computing the average rate of change for f on the interval $a \leq x \leq a + 0.001$. For example,

$$f'(-2) \approx \frac{f(-1.999) - f(-2)}{0.001} \approx 7.994$$

Using this approximation method, we can construct the following table.

Input Value x	-2.00	-1.50	-0.50	0	0.50	1.00	1.50
Approximation for $f'(x)$	7.994	2.746	-3.251	-4.000	-3.248	-0.997	2.755

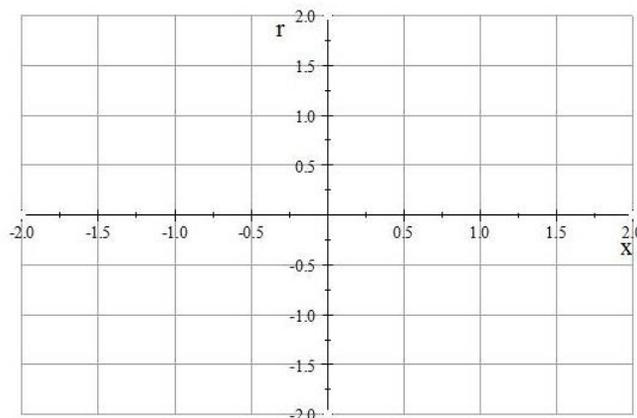
Plotting these ordered pairs and connecting them with a smooth curve gives us an approximation to the actual graph of the derivative function.



Problem 4. Use the method in Example 3 to construct a sketch of the derivative function $r = f'(x)$ for the function

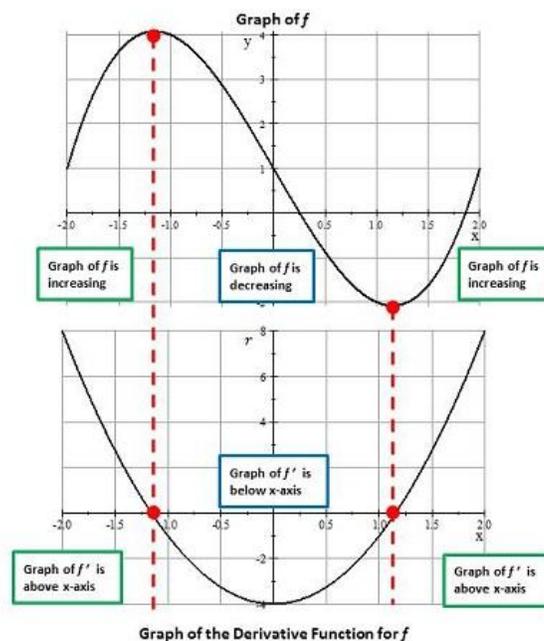
$$y = f(x) = \frac{1 + \cos(x)}{1 + x^2}$$

Input Value x	-2.00	-1.50	-0.50	-0.25	0.00	0.25	1.00	1.50	2.00
Approximation for $f'(x)$									



Problem 5. Use your sketch from Problem 4 to estimate the value of $f'(0.75)$ and then construct the approximate formula for the line tangent to the graph of the function f in Problem 4 at the point $(0.75, f(0.75))$.

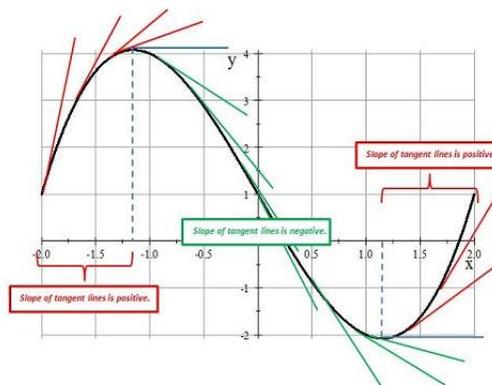
There is an important relationship between the graph of a function and the graph of its derivative function. Consider the diagram below which shows the graph of the function $y = f(x) = x^3 - 4x + 1$ along with the graph of its derivative function.



Suppose that $y = f(x)$ is a function that is locally linear at each input value in its domain. The diagram above highlights the relationship between the graph of the function f and the graph of its derivative function.

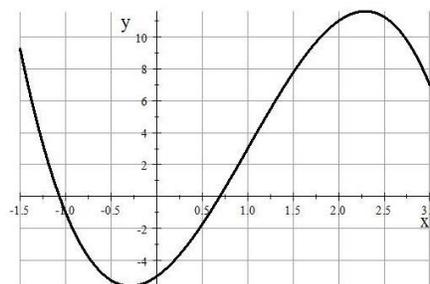
- Suppose the graph of f is decreasing on an input interval, and let $x = a$ be any input value from this interval. We can choose a value of h close enough to 0 that the average rate of change for f on the interval from $x = a$ to $x = a + h$ will be *negative*. Therefore, the value of $f'(a)$ will be negative. **Consequently, the graph of the function f' will be below the input axis on this interval.**
- Suppose the graph of f is increasing on an input interval, and let $x = a$ be any input value from this interval. We can choose a value of h close enough to 0 that the average rate of change for f on the interval from $x = a$ to $x = a + h$ will be *positive*. Therefore, the value of $f'(a)$ will be positive. **Consequently, the graph of the function f' will be above the input axis on this interval.**
- If the graph of the function f has a turning point at the input value $x = a$, then we will have $f'(a) = 0$.

We can also understand the relationship between the graph of a function f and its derivative function by thinking about tangent lines to the graph of the function f .

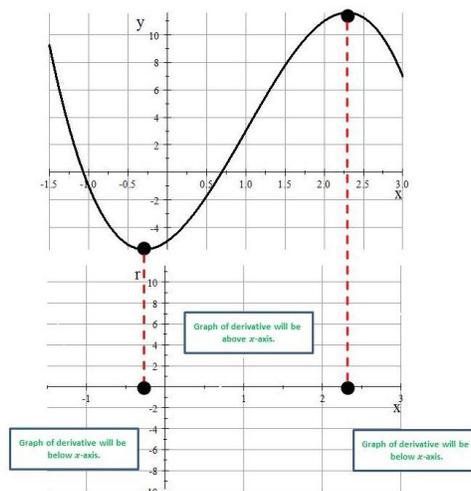


This approach is particularly helpful if we are trying to sketch the graph of the derivative function for a function $y = f(x)$ when we have only the graph of the function f to go on.

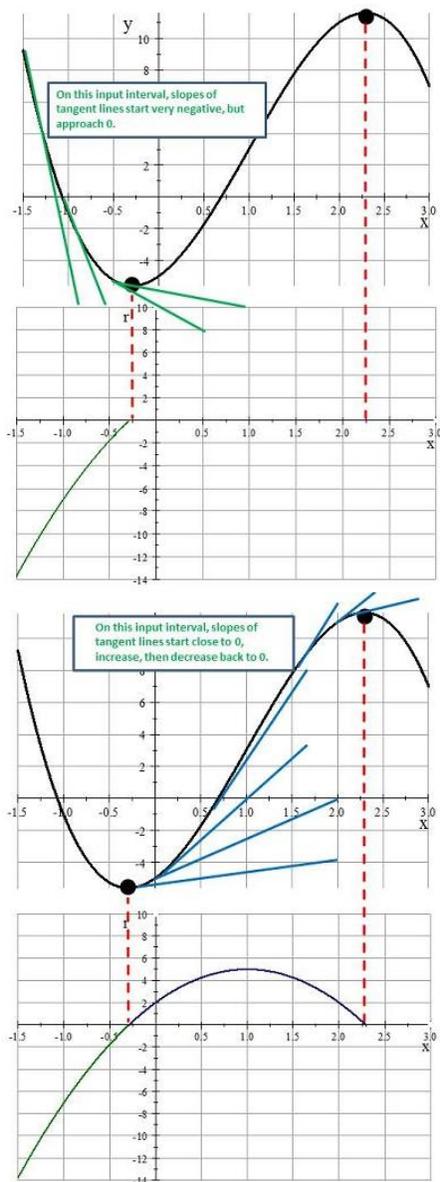
Example 2. The graph of a function $y = f(x)$ is given below. Use this graph to construct a sketch of the derivative function $r = f'(x)$.



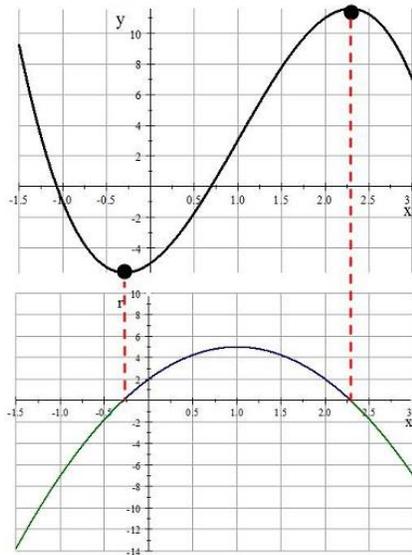
Solution. The easiest way to start would be to locate the turning points for the graph of the function f , because the input values for these points determine where the input intervals of increase and decrease for the function f occur, and they determine where the graph of the derivative function will cross the input axis.



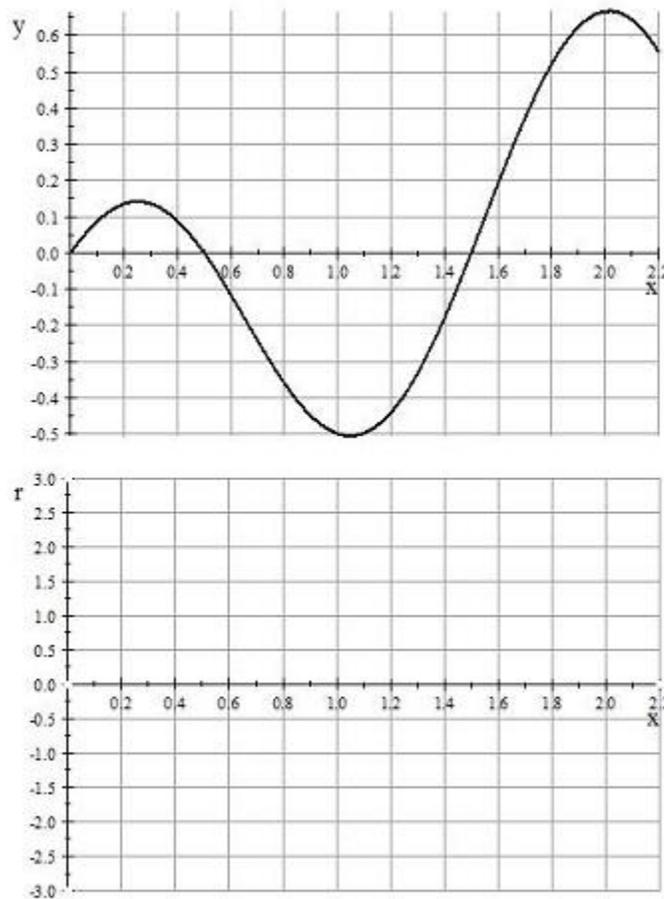
Now, we can sketch some tangent lines to the graph of the function f at various points in each interval to get an idea about how the graph of the derivative function is behaving. We could even estimate the slopes of these tangent lines to determine approximate output values of the derivative function if we wished.



On the last input interval, the slopes of tangent lines will start close to 0 (but negative), and become more negative. Consequently, the completed sketch of the derivative function will look like the following diagram.



Problem 6. The graph of a function $y = f(x)$ is shown below. Use this graph to sketch an approximation for the graph of the derivative function for f .

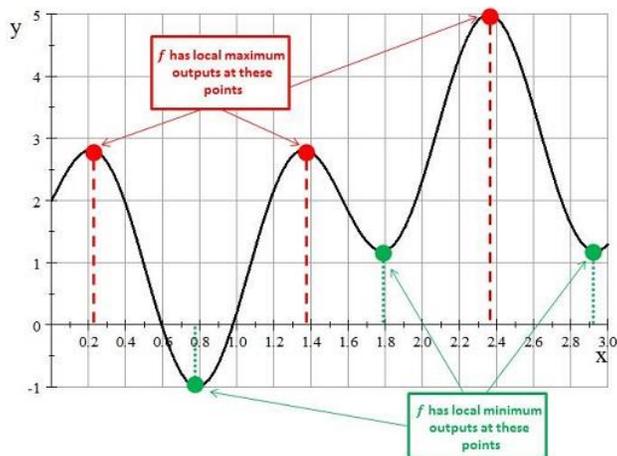


Local Extrema for a Function

Let $y = f(x)$ be a function.

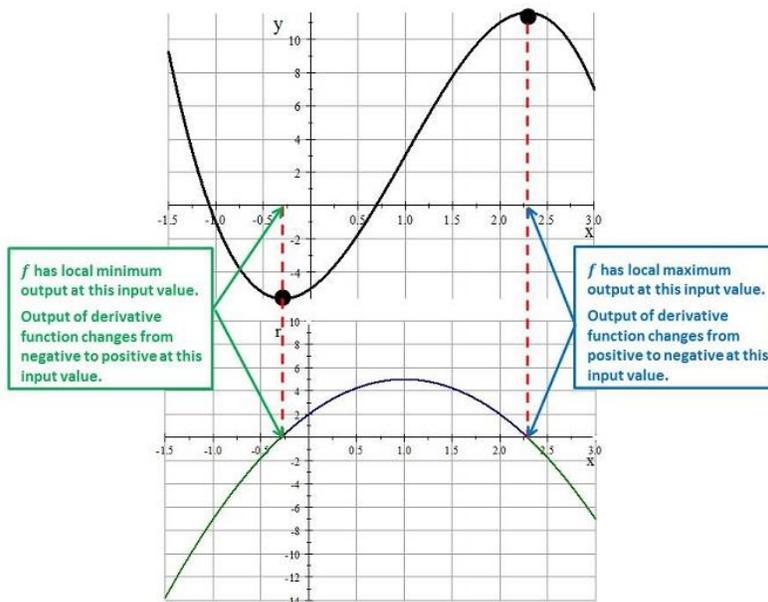
- We say that the function f has a *local maximum output* at an input value $x = a$ provided $f(a)$ is the largest output for f on some small input interval containing $x = a$.
- We say that the function f has a *local minimum output* at an input value $x = a$ provided $f(a)$ is the smallest output for f on some small input interval containing $x = a$.

We often say that the function f has a *local extremum* at an input value $x = a$ when we know that f has a local maximum or minimum output at $x = a$ but have not yet determined which one it is.



The diagram above shows the graph of a function $y = f(x)$ and identifies the input values where the function f has local extrema. For example, the function f has a local maximum output at the input values $x \approx 0.79$, $x \approx 1.39$, and $x \approx 2.39$.

There is a close connection between the derivative function for a function $y = f(x)$ and the existence of local extrema for the function f . We can see this connection by returning to the function we explored in Example 4.



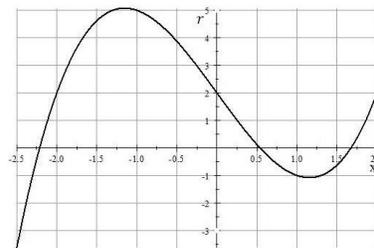
First Derivative Test

Let $y = f(x)$ be a function and let $r = f'(x)$ represent its derivative function. Suppose that $f(a)$ exists.

- If the output of f' changes from positive to negative at $x = a$, then the function f has a local maximum output at $x = a$.
- If the output of f' changes from negative to positive at $x = a$, then the function f has a local minimum output at $x = a$.

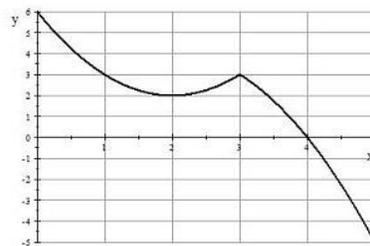
(This test assumes we are reading the graph from left to right.)

Problem 7. The graph of the *derivative function* for a function $y = f(x)$ is shown below. Based on this graph, at what input values does the function f have local maximum or local minimum output?

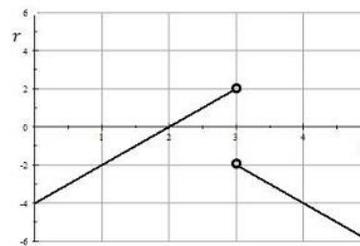


Graph of the derivative function for f

The First Derivative Test can be used even if the derivative function is undefined at $x = a$. For example, the function $y = f(x)$ shown below has a local maximum output at the input value $x = 3$, but the function f is *not* locally linear at the point $(3, f(3))$. Nonetheless, the output of the derivative function $r = f'(x)$ changes from positive to negative at $x = 3$.



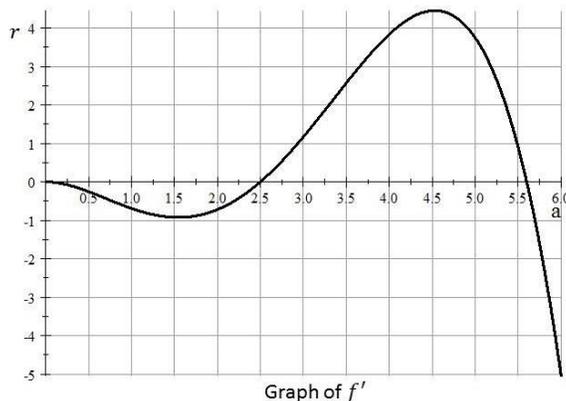
Graph of f



Graph of f'

Homework.

Problem 1. The temperature T , measured in degrees Fahrenheit, for the hull of an airplane covaries with the altitude a of the plane, measured in thousands of feet above sea level. Suppose that $T = f(a)$ is the function that gives the values of T in terms of the values of a . The diagram below shows the graph of the derivative function $r = f'(a)$ for the function f .



Part (a). The output variable for the function f' is called r . What are the units on this variable?

Part (b). What is the approximate value of $f'(3.0)$? What is the meaning of this number?

Part (c). If we assume the hull temperature of the plane is $55^\circ F$ when it is 3,000 feet above sea level, construct the formula for the tangent line to the graph of the function f at the point $(3.0, f(3.0))$.

Part (d). Based on the graph above, at what elevation above sea level will the airplane attain its maximum hull temperature?

Part (e). Based on the graph above, at what elevation above sea level will the airplane attain its minimum hull temperature?

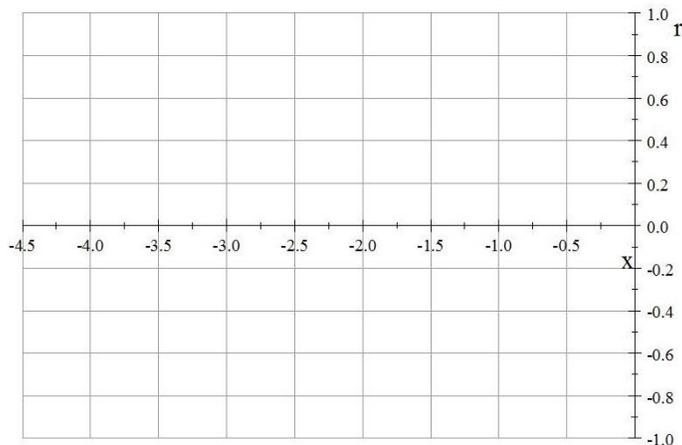
Problem 2. Consider the function $y = f(x)$ defined by the formula below.

$$f(x) = \frac{\ln(2 + \cos(x))}{2 + \sin(x)}$$

Part (a). Use the method in Example 1 to fill in the table below. (Be sure your calculator is in radian mode.)

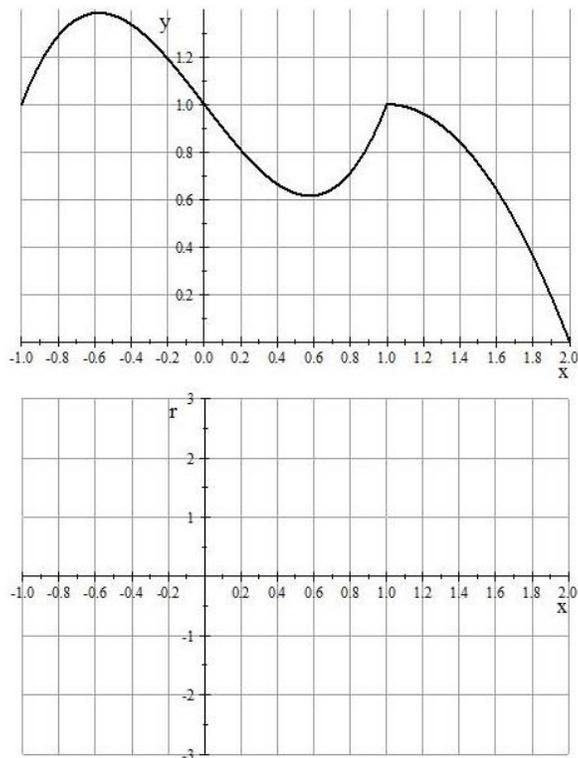
Input Value x	-4.50	-4.00	-3.50	-3.00	-2.50	-2.25	-2.00	-1.75	-1.50	-1.00	-0.50	0.00
Approximation for $f'(x)$												

Part (b). Use your table values and the grid provided to sketch the graph of the derivative function $r = f'(x)$ for the function f .



Part (c). Use the information from your graph to construct the formula for the tangent line to the graph of the function f at the point $(-3.25, f(-3.25))$.

Problem 3. The graph of a function $y = f(x)$ is shown below. Use this graph to sketch an approximation for the graph of the derivative function for f .



Answers to the Homework.**Problem 1.**

Part (a). The units are the same as those for the average rate of change --- Degrees Fahrenheit per 1000 feet of elevation above sea level.

Part (b). According to the graph, $f'(3.0) \approx 1.10$ degrees Fahrenheit per 1000 feet of elevation above sea level. This number tells us how small changes in the elevation quantity near 3,000 feet elevation will (approximately) affect changes in the temperature of the airplane. In particular, if Δa is small, we know that

$$f(3.0 + \Delta a) \approx f(3.0) + 1.10\Delta a$$

Part (c). The formula would be $T = 1.10(a - 3.0) + 55$.

Part (d). The First Derivative Test tells us the *local* maximum hull temperature will occur when the plane is about 5,600 feet above sea level. Considering the substantial *rate* of decrease in temperature for elevations greater than 5,600 feet, this local maximum is likely the greatest hull temperature the plane will experience. (We cannot be sure of this without more information.)

Part (e). The First Derivative Test tells us the *local* minimum hull temperature will occur when the plane is about 2,500 feet above sea level. However, this temperature is only a *local* minimum. Judging from the graph, it is likely that a lower hull temperature will occur at some elevation greater than 5,500 feet, since the *rate* of temperature decrease is substantial after this elevation.

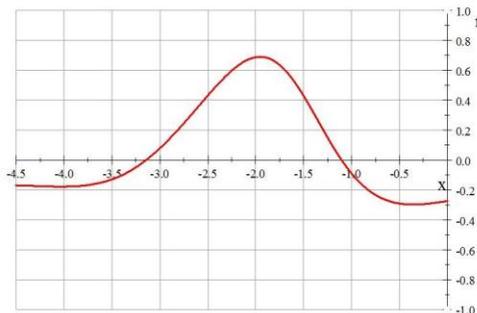
Problem 2.

Part (a). The data in the table below was obtained using the approximation

$$f'(x) \approx \frac{f(x + 0.001) - f(x)}{0.001}$$

Input Value x	-4.50	-4.00	-3.50	-3.00	-2.50	-2.25	-2.00	-1.75	-1.50	-1.00	-0.50	0.00
Approximation for $f'(x)$	-0.17	-0.178	-0.13	0.078	0.431	0.598	0.687	0.635	0.429	-0.09	-0.292	-0.275

Part (b). Here is a sketch of the graph for the function $r = f'(x)$. Your sketch may vary slightly.



Part (c). First, note that $f(-3.25) \approx 0.00278$. Based on the graph above, we see that $f'(-3.25) \approx -0.10$. Therefore, the formula for the tangent line to the graph will be

$$y = T(x) \approx -0.10(x + 3.25) + 0.00278$$

Problem 3. Here is the approximate sketch of the derivative graph. Your graph may vary slightly.

