We have already seen that the derivative function encodes a considerable amount of information about the graph of the function it is created from. In particular, we have already worked with the following observations (albeit only with graphs).

First Derivative Test

Let $y = f(x)$ be a function and let $r = f'(x)$ represent its derivative function. Suppose that $f(a)$ exists.

- If the output of f' changes from positive to negative at $x = a$, then the function f has a local maximum output at $x = a$.
- If the output of f' changes from negative to positive at $x = a$, then the function f has a local minimum output at $x = a$.

(This test assumes we are reading the graph from left to right.)

The First Derivative Test applies to any function that we can differentiate. This means that, if we can construct the derivative function for the *derivative* of some function f, then we can apply the First Derivative Test to the function f' … and the same conclusions apply.

Second Derivative

Let $y = f(x)$ be a function. The *second derivative* function for the function f is defined to be the derivative function for the derivative function of f. The second derivative function provides the instantaneous rate of change for the derivative function at any input value. The second derivative function is commonly denoted by

$$a = f''(x) \qquad \text{OR} \qquad a = \frac{d}{dx}\left[\frac{df}{dx}\right] \qquad \text{OR} \qquad a = \frac{d^2f}{dx^2}$$

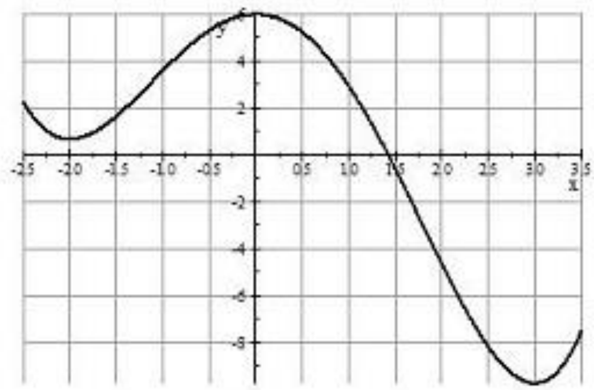
For example, whenever the second derivative function f'' has positive output, we know that the *derivative function* f' is increasing; and whenever the output of f'' changes from positive to negative at some input value $x = a$, we know that the *derivative function* f' has a local maximum output at the input value $x = a$. In this investigation, we will explore what implications this holds for the original function f.

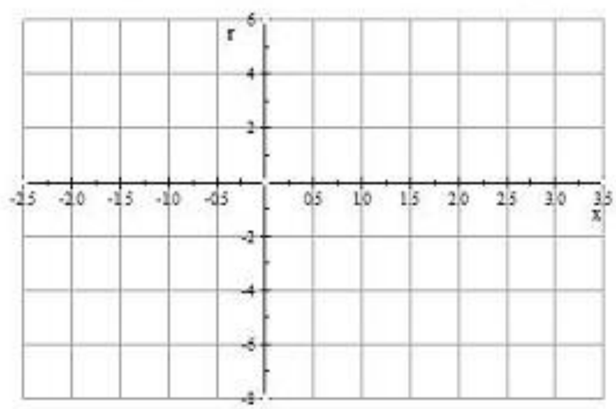
Problem 1. Suppose that $y = f(x)$ is *twice differentiable* (that is, suppose that $a = f''(x)$ exists) on some input interval.

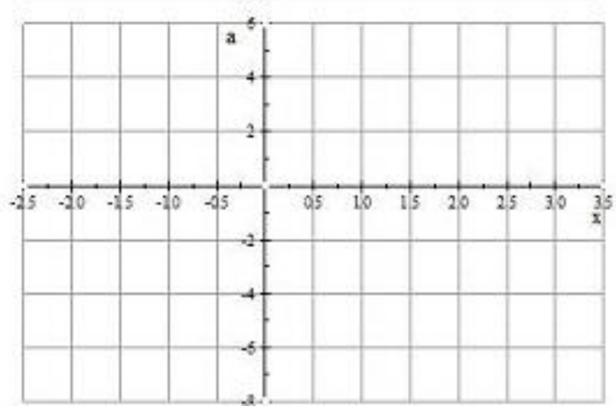
Part (a). Explain why the function $r = f'(x)$ must also exist on this input interval.

Part (b). Construct the formula that gives the limit definition of the function $a = f''(x)$.

Problem 2. The diagram below shows the graph of a function $y = f(x)$. On the first grid, sketch the graph of the derivative function $r = f'(x)$ for the function f. Use your sketch of the derivative function f' to sketch the second derivative function $a = f''(x)$.

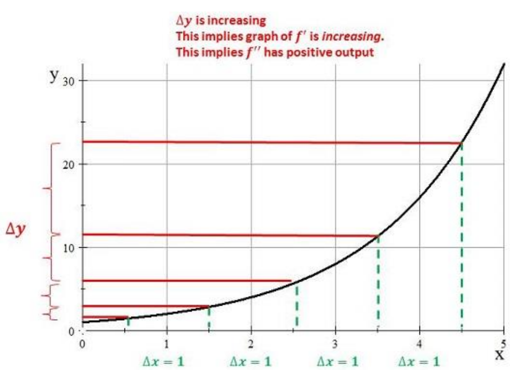


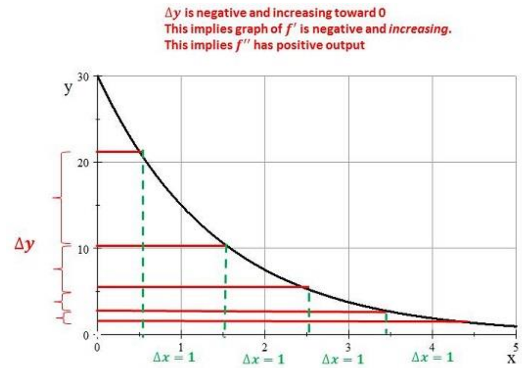




Problem 3. There is a relationship between the graph of the second derivative function and the input intervals where the graph of f is concave up or concave down. Describe what this relationship is.

The output of the second derivative function for a function $y = f(x)$ tells us the instantaneous rate of change *of the rate of change* for the function f. Because of this, the second derivative gives us information about the *concavity* of the function f.





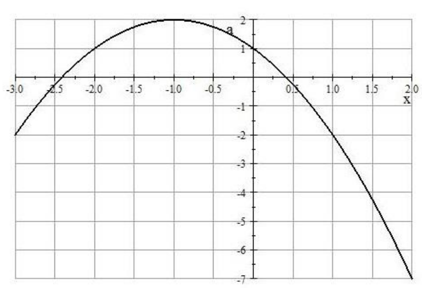
- If a function $y = f(x)$ is increasing at an increasing rate, then the second derivative function will have positive output, and the graph of the function f will be concave up.
- If a function $y = f(x)$ is decreasing at an increasing rate, then the second derivative function will have positive output, and the graph of the function f will be concave up.

- If a function $y = f(x)$ is increasing at a decreasing rate, then the second derivative function will have negative output, and the graph of f will be concave down.
- If a function $y = f(x)$ is decreasing at a decreasing rate, then the second derivative function will have negative output, and the graph of f will be concave down.

The output sign of the second derivative function on an input interval signals the concavity of the function on that input interval.

- If the output of the second derivative function is positive on an input interval, then the function is concave up on that input interval.
- If the output of the second derivative function is negative on an input interval, then the function is concave down on that input interval.
- If the output of the second derivative function changes sign at some input value $x = a$, then the function f has an *inflection point* at $x = a$. (That is, $x = a$ is an input value where the concavity of the function f changes.

Problem 4. The diagram below shows the graph of the second derivative function for a function $y = f(x)$. On which input intervals will the graph of the function f be concave up? On which input intervals will the graph of the function f be concave down? At what input values will the function f have an inflection point?



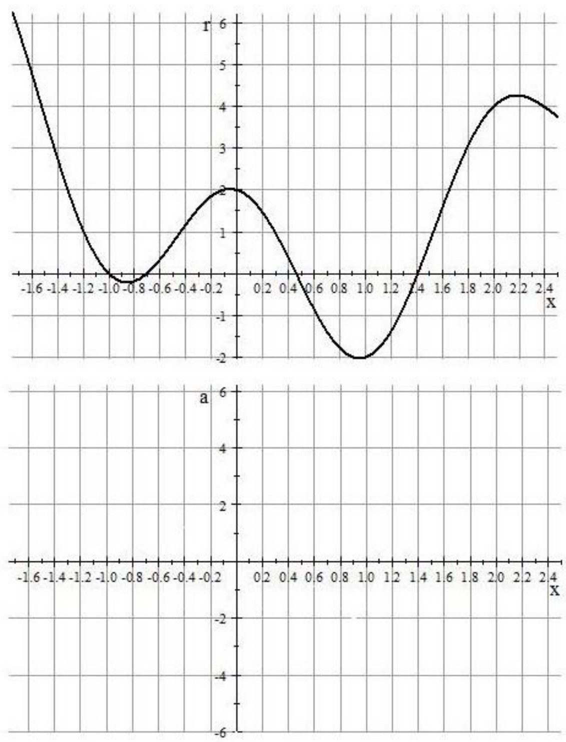
Graph of Second Derivative Function

Problem 5. The graph below shows the derivative function for a function $y = f'(x)$.

Part (a). Based on this graph, at which input values does the function f have inflection points? How did you decide?

Part (b). On the grid provided, construct a sketch of the graph for the second derivative function $a = f''(x)$.

Graph of f'

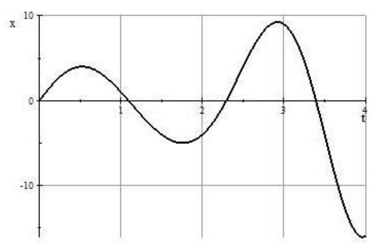


Problem 6. Justo's cat, Mr. Hipsickles, is running back and forth chasing a fly. Let x represent the values of the quantity "Mr. Hipsickles' distance to the right of Justo, measured in feet" and let t represent the values of the quantity "the number of seconds passed since Mr. Hipsickles started chasing the fly." Let $x = f(t)$ represent the function that gives the values of x in terms of the values of t.

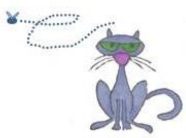
Part (a). If we let the first derivative function for f be represented by $r = f'(t)$, then what units will the output variable r have? What quantity values does r represent in this process?

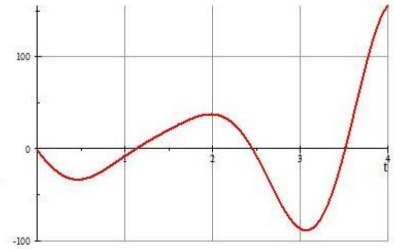
Part (b). If we let the second derivative function for f be represented by $a = f''(t)$, then what units will the output variable a have? What quantity values does a represent in this process?

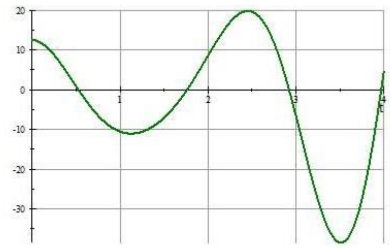
Part (c). The leftmost graph below shows the function $x = f(t)$. One of the other graphs shows the function $r = f'(t)$, and the other graph *may or may not* show the function $a = f''(t)$. Identify the graphs, and explain how you decided.



Graph of the Function f

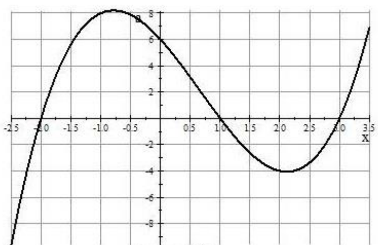




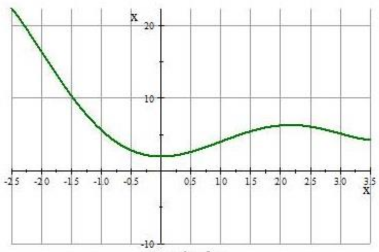


Homework.

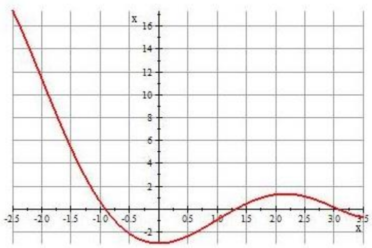
Problem 1. The leftmost graph below shows the second derivative function for some function $y = f(x)$. Could either of the graphs on the right represent the original function f? Justify your answer.



Graph of f''



Graph of g

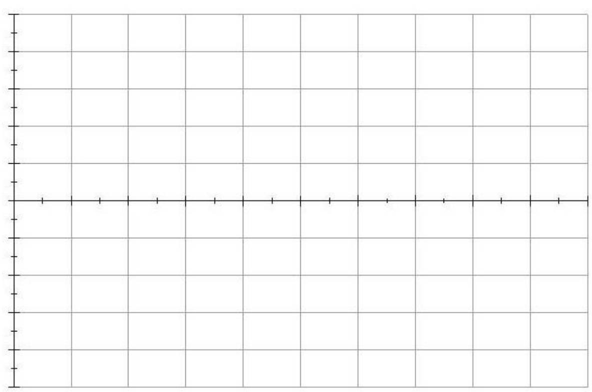


Graph of h

Problem 2. Tox, Inc. was recently cited for dumping polluted water into Lake Lotta Watta, causing a decline in the number of blue-finned mud twaddlers present in the lake. Tox, Inc. defended itself by stating that the population of mud twaddlers in the lake is "decreasing at a decreasing rate."

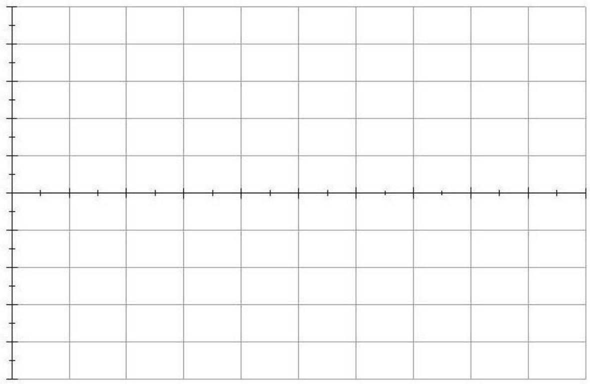
Part (a). Is this really a defense? Explain your thinking.

Part (b). Use the grid provided to sketch a graph that could give the population of blue-finned mud twaddlers as a function of time. Be sure to correctly identify the quantities used and label your axes accordingly.



Problem 3. On the grid provided, sketch the graph of a function $y = f(x)$ that satisfies the following six conditions. There are many possible answers. Be sure to label your axes.

- ... On the input interval $0 \le x < 3$ we know $f'(x) < 0$.
- ... We know $f'(3) = 0$.
- ... On the input interval $3 < x \le 4$ we know $f'(x) > 0$.
- ... On the input interval $0 \le x < 1.5$ we know $f''(x) < 0$.
- ... We know $f''(1.5) = 0$.
- ... On the input interval $1.5 < x \le 4$ we know $f''(x) > 0$.

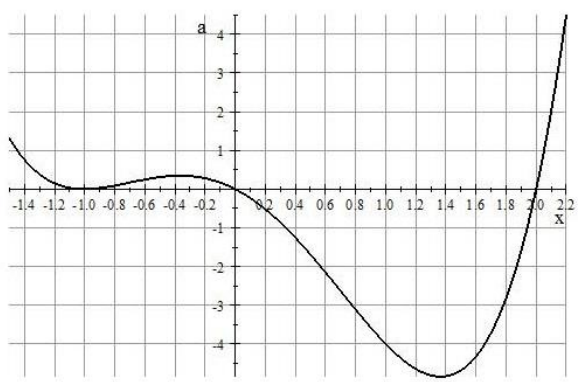


Problem 4. The diagram below shows the graph of the second derivative function for a function $y = f(x)$.

Part (a). At which input values does the function f have inflection points?

Part (b). On which input intervals is the function f concave up?

Part (c). On which input intervals is the function f concave down?



Graph of f''

Answers to the Homework.

Problem 1. The graph of the second derivative tells us the following about the function f.

- ... The graph of the function f must be concave down on the input intervals $x < -1$ and $1 < x < 3$.
- ... The graph of the function f must be concave up on the input intervals $-1 < x < 1$ and $3 < x$.
- ... The graph of the function f must have an inflection point at $x = -1$, $x = 1$, and $x = 3$.

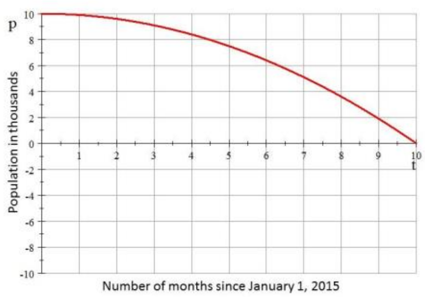
Inspection shows us that *both* graphs on the right satisfy these conditions (although the concavity of both graphs is debatable on the input interval $-2 < x < -1$). Therefore, the graph on the left *could* serve as the second derivative for both of the functions whose graphs are shown on the right.

Problem 2.

Part (a). To say that the population is decreasing at a decreasing rate implies that the population graph is decreasing *and* concave down. This is not really a defense, unless you want the mud twaddlers to go extinct.

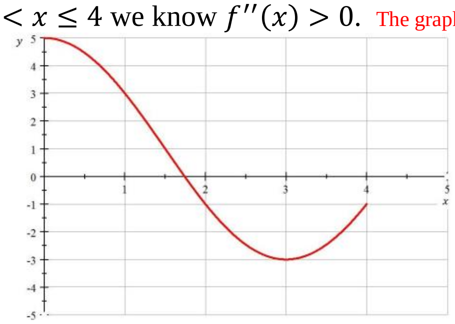
Part (b). Here is one rather pessimistic example.

Let t represent the number of months passed since January 1, 2015, and let p represent the population of mud twaddlers present in the lake, measured in thousands.



Problem 3. The graph shown below is one of many that satisfy the criteria laid out in this problem.

- ..... On the input interval $0 \leq x < 3$ we know $f'(x) < 0$. The graph of f is decreasing on this input interval.
- ... We know $f'(3) = 0$.
- ... On the input interval $3 < x \leq 4$ we know $f'(x) > 0$. The graph of f is increasing on this input interval.
- ... On the input interval $0 \leq x < 1.5$ we know $f''(x) < 0$. The graph of f is concave down on this input interval.
- ... We know $f''(1.5) = 0$.
- ... On the input interval $1.5 < x \leq 4$ we know $f''(x) > 0$. The graph of f is concave up on this input interval.



Problem 4.

Part (a). The function f will have inflection points at $x = 0$ and at $x = 2$. The function will *not* have an inflection point at $x = -1$ because the output sign of the second derivative function does not change.

Part (b). The function f will be concave up on the input intervals $x < -1$, $-1 < x < 0$, and $2 < x$.

Part (c). The function will be concave down on the input interval $0 < x < 2$.