In this investigation, we will introduce what is arguably the most important insight that calculus gives us regarding covariation. It may not seem like much at first, but this result underlies virtually every technique used in Calculus I. Let's start with a problem.

The farm-to-market trucking company Haulin' Oats LLC is very proud of its safety record and has a strict no-tolerance policy regarding speed limits. The company requires all of its employees to keep their trucking speeds under 65 mph. The company uses a satellite tracking device on every truck that records the distance traveled from the pickup site as well as the truck velocity.

Sathya picked up a load of barley at a warehouse in Kansas City and delivered it to the central warehouse west of Topeka exactly half an hour later as recorded by the company foreman. His tracking device malfunctioned and did not record his velocity; however, it did manage to record his distance from the warehouse in Kansas City. The diagram below shows Sathya's distance s (measured in miles) from the warehouse in Kansas City as a function of the number t of hours since he left the warehouse.

Barbara, the human resources manager for Haulin' Oats, looked at the graph and fired Sathya on the spot. Sathya appealed the dismissal, claiming that, while his average speed on the time interval $0 \leq t \leq 0.5$ hours did exceed the maximum allowed velocity, his *instantaneous velocity* never did.

Problem 1. What was Sathya's average velocity on the time interval $0 \leq t \leq 0.5$ hours?

Problem 2. Do you think Sathya's instantaneous velocity ever exceeded the 65 mph requirement? Justify your thinking using the graph.

Problem 3. Let $s = f(t)$ be the function that gives the values of s in terms of the values of t.

Part (a). On the graph provided, draw the secant line for the function $s = f(t)$ on the time interval $0 \leq t \leq 0.5$ hours.

Part (b). Are there any time values where the tangent line to the graph of f is parallel to this secant line?

Part (c). Explain how your answer to Part (b) allows you to decide whether Sathya should be reinstated.

> ***Mean Value Theorem***
>
> Suppose that $y = f(x)$ is a function that is continuous on the closed input interval $a \leq x \leq b$ and differentiable on the open input interval $a < x < b$. There will always be at least one input value $x = c$ where the tangent line to the graph of the function f is parallel to the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.

Problem 4. Sathya countered by claiming that the tracking device's data could not be correct, citing as evidence the graph on the input interval $0 \leq t \leq 0.10$ hours and the fact that his truck is incapable of velocities greater than 100 mph. Do you agree with Sathya? Why or why not?

Problem 5. Barbara stated that the actual data is irrelevant --- Sathya traveled 35 miles in thirty minutes. His instantaneous velocity *must* have been at least 70 mph at some time. Does the Mean Value Theorem support Barbara's conclusion? Explain.

The Mean Value Theorem tells us something profound about the derivative function. Suppose that $y = f(x)$ is a differentiable function, and let $r = f'(x)$ be its derivative function. The Mean Value Theorem tells us that the derivative function *contains all of the average rate of change information for the function.*

To understand what this means, choose any distinct input values $x = a$ and $x = b$ in the domain of the function f, and suppose $a < b$. We can construct the average rate of change for the function f on the input interval $a \leq x \leq b$. The average rate of change will be

$$m = \frac{f(b) - f(a)}{b - a}$$

This values is the constant rate of change for the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$; therefore, the Mean Value Theorem tells us that there is *some* input value $x = c$ between $x = a$ and $x = b$ where $f'(c) = m$. Consequently, any average rate of change we can compute for the function f is produced as output from the derivative function. Furthermore, this output will occur somewhere in the input interval we used in computing the average rate of change.

The importance of the Mean Value Theorem is mostly theoretical. It is the Mean Value Theorem that justifies the First Derivative Test, for example. The Mean Value Theorem also justifies the claim that a concave-up function will have a positive second derivative function (whenever the second derivative function exists).

We will leave this investigation by looking at one example of how the Mean Value Theorem underpins much of what we use in calculus.

Positive Output from Derivative Implies Increasing Function

Suppose that $y = f(x)$ is a function that differentiable on some input interval $a < x < b$. If we know that $f'(x) > 0$ for all x values in this input interval, then the graph of f must be increasing on this input interval.

Proof. If we want to show that the graph of f is increasing, we will need to show that $f(u) < f(v)$ whenever we have $u < v$. To this end, suppose that $a < u < v < b$. By assumption, we know that f is differentiable on the open interval $u < x < v$. Since a differentiable function must be continuous, and since f is assumed to be differentiable on the input interval $a < x < b$, we may conclude that f is also continuous on the closed interval $u \leq x \leq v$. The Mean Value Theorem therefore tells us that there exists some value $x = c$ between $x = u$ and $x = v$ such that

$$f'(c) = \frac{f(v) - f(u)}{v - u}$$

Now, we have assumed that the output of f' is positive; and since we have assumed $u < v$, we also know that $v - u > 0$. Consequently, we know $0 < (v - u)f'(c)$; and it follows that

$$f'(c) = \frac{f(v) - f(u)}{v - u} \quad \Rightarrow \quad (v - u)f'(c) = f(v) - f(u) \quad \Rightarrow \quad 0 < f(v) - f(u)$$

Thus, $f(u) < f(v)$ as desired.