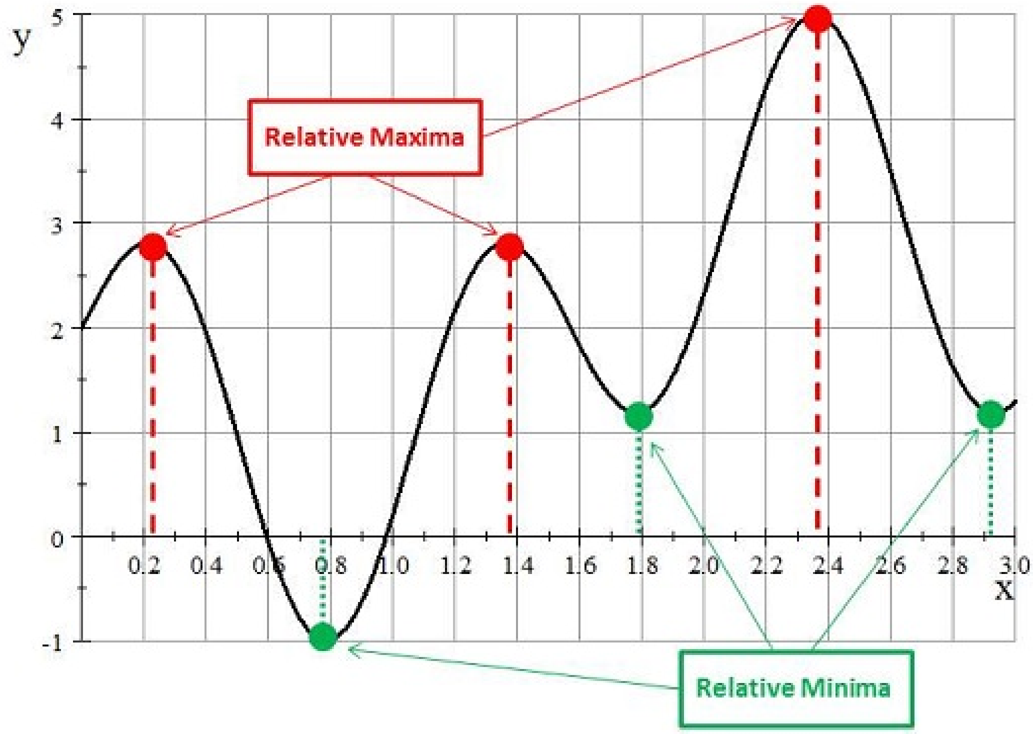
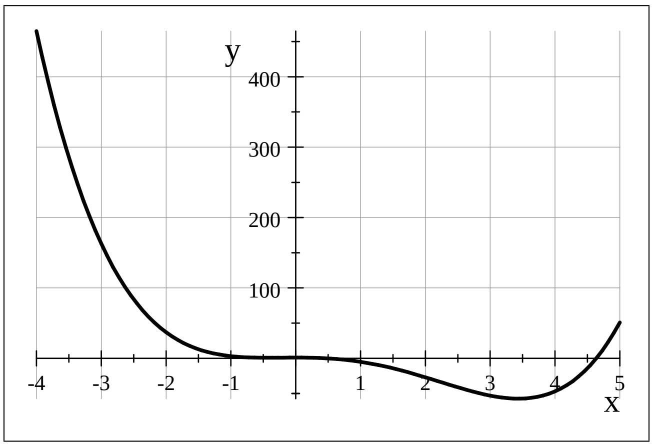
# Classifying Relative Extrema for Functions

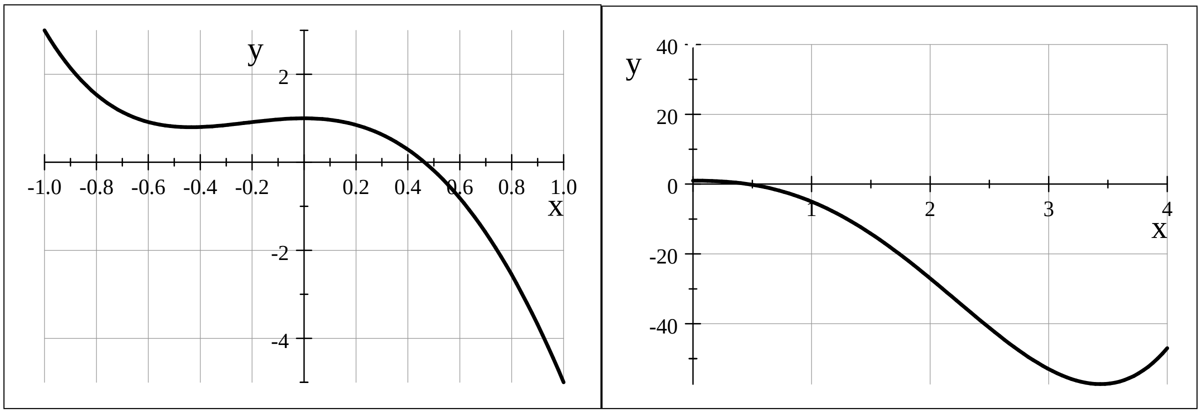
If we are fortunate enough to have the graph of a function $f$, then locating the relative extrema and classifying them as relative maxima or relative minima is *usually* straightforward.




However, this is not always the case. Consider the function $f(x) = x^4 - 4x^3 - 3x^2 + 1$. The graph of this function is not particularly helpful when it comes to identifying relative extrema for $f$. For example, here is the graph of $f$ on a s reasonable viewing interval




From the graph, it appears that there is a relative minimum somewhere between $x = 3$ and $x = 4$, but it is difficult to tell what, if anything, is happening near $x = 0$. If we graph the function on different scales near $x = 0$ and near $x = 3$, we can get a better idea.




From these graphs, we see that $f$ has at least three turning points (there could be more to the right of $x = 3$ or to the left of $x = -1$ for all we know). However, it is still not clear exactly where these turning points occur; and it is certainly not obvious where the inflection points lie (although one does appear to be close to $x = 2$). We can use calculus to avoid these problems.

The process for locating and classifying the relative extrema for $f$ begins by first locating its critical numbers, since the relative extrema must occur at critical numbers for $f$. Note that

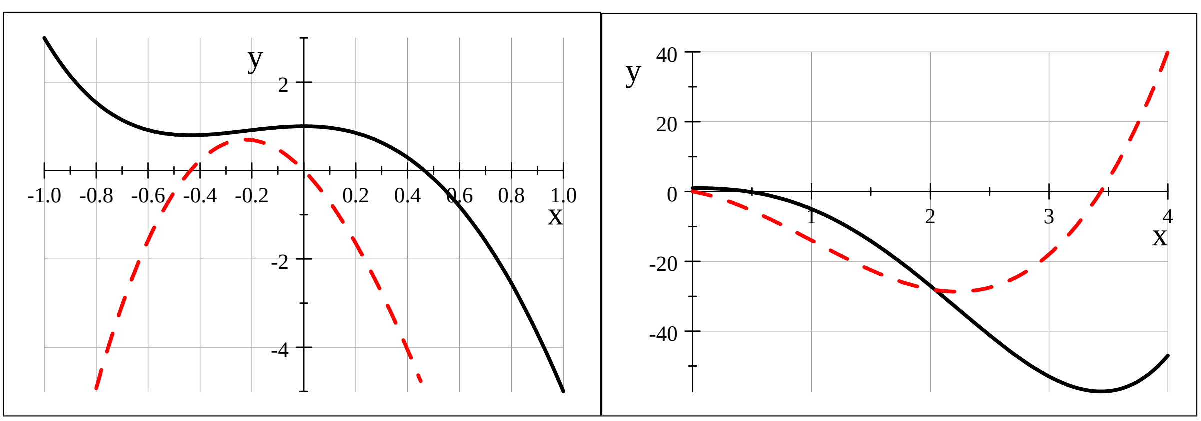
$$f'(x) = 4x^3 - 12x^2 - 6x = 2x(2x^2 - 6x - 3)$$

Since there is no division in the formula for $f'$, the only critical numbers for $f$ will occur at values of $x$ that make $f'(x) = 0$. Observe that

$$\begin{aligned} f'(x) &= 0 \Longrightarrow 2x = 0 \quad \text{or} \quad 2x^2 - 6x - 3 = 0 \\ &\Longrightarrow \quad x = 0 \quad \text{or} \quad x = \frac{3 \pm \sqrt{15}}{2} \end{aligned}$$

Hence, $f$ has exactly three critical numbers, namely $x \approx -.44$, $x = 0$, and $x \approx 3.44$. Now, if we look at the graphs above, we can see $f$ must have a relative minimum output at $x \approx -.44$, a relative maximum output at $x = 0$, and another relative minimum output at $x \approx 3.44$.

Now, suppose we *don't* have the graphs for $f$. We can still classify the relative extrema for $f$ if we think about how the derivative is behaving *in between* the critical numbers we found. The graphs below show the function $f$ along with its derivative function on intervals close to the critical numbers.




In Chapter 2, we learned how to tell whether a critical number for a function corresponds to a relative maximum or relative minimum output for $f$ — the derivative function will *change sign* as you cross over the critical number from left to right. Notice that the sign of the derivative graph *remains the same in between the critical numbers.* For example, between $x = 0$ and $x \approx 3.44$, the graph of the derivative function is below the $x$-axis (its sign is negative).

Therefore, if we do not have the graph for $f$, we can check the sign of the derivative function in between two consecutive critical numbers simply by taking a single "test value" chosen between them and plugging that value into the derivative function.

- If the output of the derivative function for a function $f$ has a negative sign everywhere in between two input values, then the graph of $f$ is decreasing between these input values.
- If the output of the derivative function for a function $f$ has a positive sign everywhere in between two input values, then the graph of $f$ is increasing between these input values.

**Example 1** *Use "test values" to classify the relative extrema for the function* $f(x) = x^4 - 4x^3 - 3x^2 + 1$.

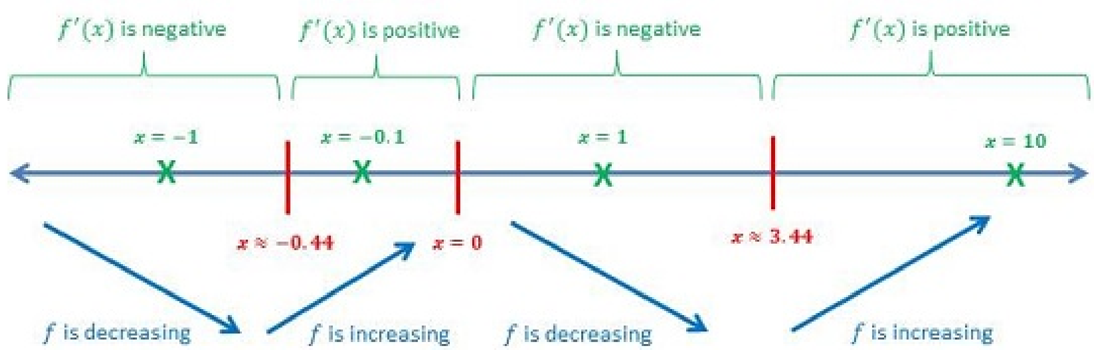
**Solution.** We already know that $f'(x) = 4x^3 - 12x^2 - 6x$, and we already know the critical numbers for $f$:

$$x \approx -0.44 \qquad x = 0 \qquad x \approx 3.44$$

These numbers divide the $x$-axis into four subsets — the ray $(-\infty, -0.44]$, the interval $[-0.44, 0]$, the interval $[0, 3.44]$, and the ray $[3.44, +\infty)$. The derivative function is only able to change sign at the endpoints of these sets — it MUST keep the same sign inside of each set. Therefore, to determine what sign the derivative function will have inside each set, we only need to check the derivative function output at a single "test point" chosen from inside each set. We can choose any "test point" we like, as long as it is not an endpoint.

- From the ray $(-\infty, -0.44]$, select $x = -1$ and observe that $f'(-0.1) = -6 < 0$. This tells us that the output of $f'$ is NEGATIVE everywhere in the set $-\infty < x < -0.44$.
- From the interval $[-0.44, 0]$, select $x = -0.1$ and observe that $f'(-0.1) = 0.476 > 0$. This tells us that the output of $f'$ is POSITIVE everywhere in the set $-0.44 < x < 0$.
- From the interval $[0, 3.44]$, select $x = 1$ and observe that $f'(1) = -5 < 0$. This tells us that the output of $f'$ is NEGATIVE everywhere in the set $0 < x < 3.44$.
- From the ray $3.44, +\infty)$, select $x = 10$ and observe that $f'(10) = 2740 > 0$. This tells us that the output of $f'$ is POSITIVE everywhere in the set $3.44 < x < +\infty$.

We can use a number line to help us pair the what we have just learned with what we know about how the behavior of the derivative function influences the behavior of the function.




We can now see that $f$ has relative minimum output at $x \approx -0.44$ and $x \approx 3.44$. We also see that $f$ has relative maximum output at $x = 0$.

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The previous example illustrates how we can use the so-called *First Derivative Test* to classify relative extrema for a function.

FIRST DERIVATIVE TEST

- Suppose that $x = a$ is a critical number for a function $f$ whose derivative exists on some interval containing $x = a$ (except possibly at $x = a$ itself).
  - If there exists a set $I = b < x < a$ and a set $J = a < x < c$ such that the output of $f'$ is negative on $I$ and positive on $J$, then the function $f$ has a relative minimum output at $x = a$.
  - If there exists a set $I = b < x < a$ and a set $J = a < x < c$ such that the output of $f'$ is positive on $I$ and negative on $J$, then the function $f$ has a relative maximum output at $x = a$.
  - If the sign of the derivative does not change at $x = a$, then $f$ does not have a relative extremum at $x = a$.

**Problem 1.** Consider the function $f(x) = -2x^3 + 15x^2 - 24x + 1$.

**Part (a):** Differentiate the function $f$.

**Part (b):** Identify all of the critical numbers for $f$.

**Part (c):** Follow Example 1 to classify the relative extrema for $f$.

**Problem 2.** Consider the function $f(x) = (2x - 1)^3$.

**Part (a):** Differentiate the function $f$.

**Part (b):** Identify all of the critical numbers for $f$.

**Part (c):** Follow Example 1 to classify the relative extrema for $f$.

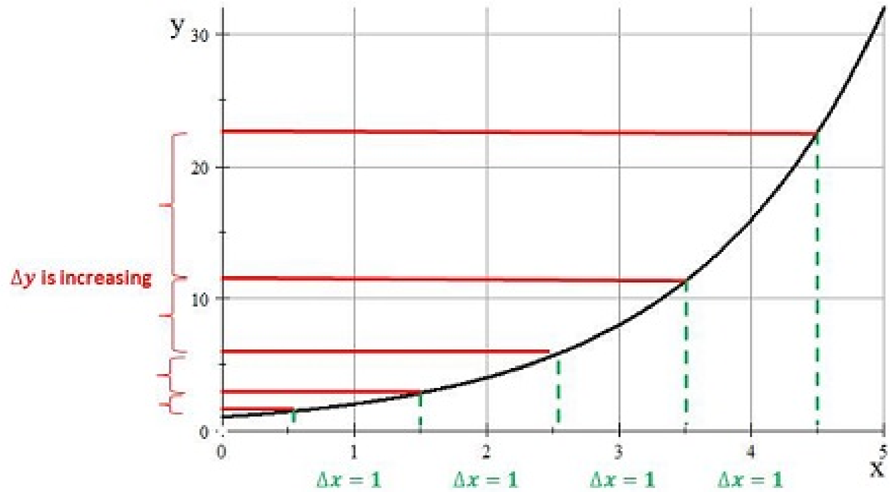
A function $f$ is said to be *concave up* on an input interval $a < x < b$ provided the graph of $f$ is "bending up" on this interval. The function is *concave down* on an input interval provided its graph is "bending down" on the interval.

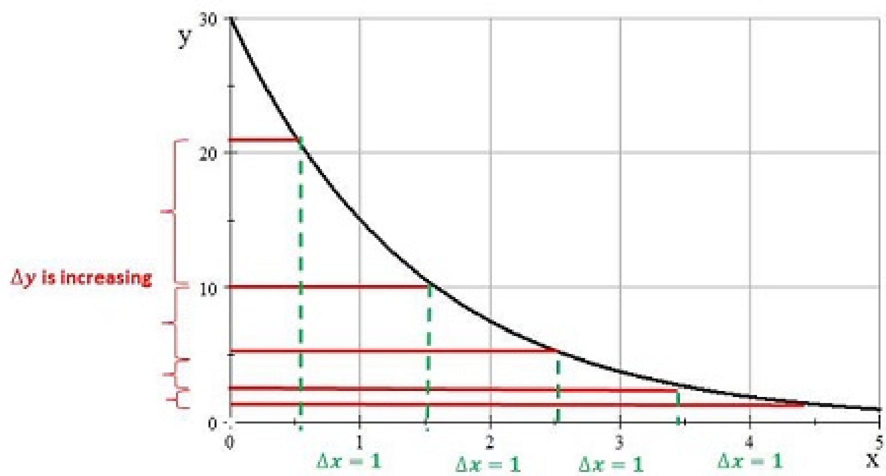
It turns out that there is a connection between the concavity of a function $f$ on an interval and the *second* derivative function for $f$. Remember that the second derivative of a function $f$ is defined to be the derivative of the *derivative function* for $f$. This means that the sign of the second derivative tells us whether or not the derivative for $f$ is itself increasing or decreasing.

- Suppose that $f$ is a twice-differentiable function. If $f''$ is positive on an interval, then the graph of the derivative $f'$ is increasing on that interval.

- Suppose that $f$ is a twice-differentiable function. If $f''$ is negative on an interval, then the graph of the derivative $f'$ is decreasing on that interval.

Now, suppose that the graph of the derivative function for $f$ is increasing. This means that the rate of change for the function $f$ is increasing. This fact forces the graph of $f$ to *bend upward.* In other words, the graph of $f$ must be concave up.




This observation tells us that when $f$ is concave up, its *derivative function* must be increasing. Since the *second derivative* of the function $f$ is the derivative of the derivative function for $f$, we have the following result:

- Suppose that $f$ is a twice-differentiable function. If $f''$ is positive on an input interval for $f$, then the graph of $f$ is concave up on this interval.
- Suppose that $f$ is a twice-differentiable function. If $f''$ is negative on an input interval for $f$, then the graph of $f$ is concave down on this interval.

Since an inflection point for $f$ is a point on the graph of $f$ where the concavity of $f$ changes, it follows that inflection points for $f$ must occur at critical numbers for $f'$ — that is, at input values where $f''$ is either undefined or has output 0.

**Example 2** *Where are the inflection points for $f(x) = x^4 - 4x^3 - 3x^2 + 1$ located?*

**Solution.** We start by identifying the critical numbers for $f'$. Observe that since $f'(x) = 4x^3 - 12x^2 - 6x$, we know

$$f''(x) = 12x^2 - 24x - 6$$

There is no division in the formula for the second derivative, so the only critical numbers for $f'$ will satisfy the equation $f''(x) = 0$. Now

$$\begin{aligned} 12x^2 - 24x - 6 = 0 &\implies 2x^2 - 4x - 1 = 0 \\ &\implies x = \frac{4 \pm \sqrt{24}}{4} \qquad \text{Apply Quadratic Formula} \\ &\implies x = 1 \pm \frac{\sqrt{6}}{2} \end{aligned}$$

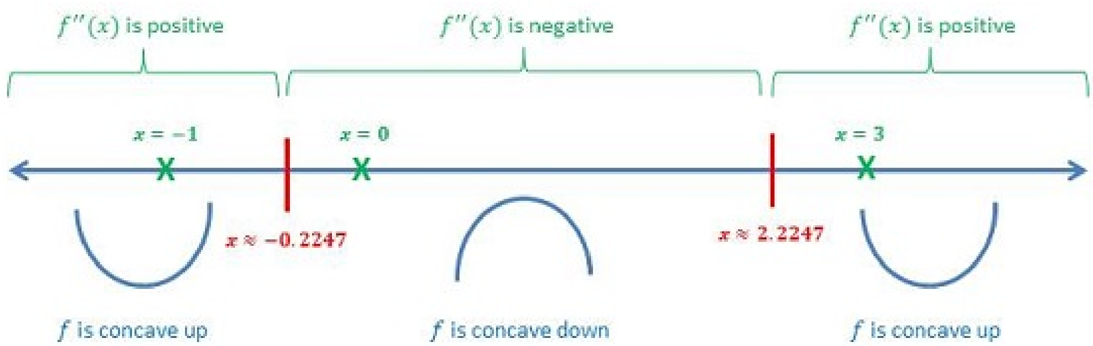
Therefore, we know that the critical numbers for $f'$ will be $x \approx 2.2247$ and $x \approx -0.2247$. We can determine whether or not these critical numbers correspond to inflection points for $f$ by checking the *sign of the second derivative* on either side of these numbers. We will need three "test values" for this check — one chosen from inside the ray $(-\infty, -0.2247]$, one chosen from the interval $[-0.2247, 2.2247]$, and one chosen from the ray $[2.2247, +\infty)$.

- From the ray $(-\infty, -0.2247]$ choose the test value $x = -1$. Observe that $f''(-1) = 26 > 0$. Hence, $f''$ is POSITIVE everywhere in $-\infty < x < -0.2247$.

- From the interval $[-0.2247, 2.2247]$ choose the test value $x = 0$. Observe that $f''(0) = -6 < 0$. Hence, $f''$ is NEGATIVE everywhere in $-0.2247 < x < 2.2247$.

- From the ray $[2.2247, +\infty)$ choose the test value $x = 3$. Observe that $f''(3) = 30 > 0$. Hence, $f''$ is POSITIVE everywhere in $2.2247 < x < +\infty$.

We can use a number line to help us pair the what we have just learned with what we know about how the behavior of the second derivative function influences the behavior of the function.




Since the concavity of $f$ changes at both critical numbers for $f'$, we may conclude that $f$ has two inflection points — one occuring when $x \approx -0.2247$ and the other occuring when $x \approx 2.2247$.

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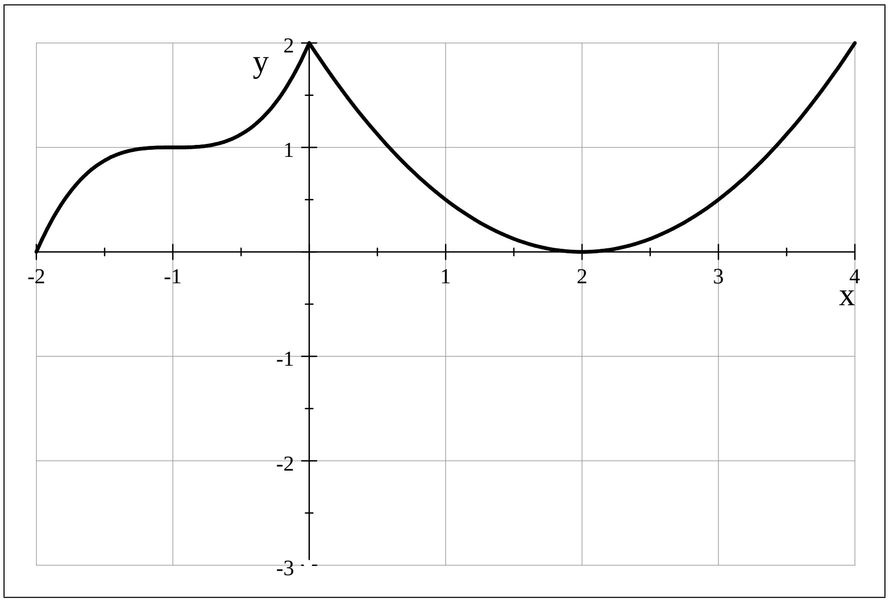
**Problem 3.** Where do the inflection points for $f(x) = (2x - 1)^3$ occur?

**Problem 4.** Where do the inflection points for $f(x) = -2x^3 + 15x^2 - 24x + 1$ occur?

It is sometimes also possible to use the second derivative of a function to classify relative extrema for a function $f$. The key is to notice that the graph of a function $f$ USUALLY will be *concave down* at a relative maximum output, and USUALLY will be *concave up* at a relative mimimum output.

We add the caveat "usually" to this observation, because it is not always true, as the following diagram shows.




This observation will be true *as long as $f$ is twice differentiable.*

SECOND DERIVATIVE TEST

- Let $f$ be a twice-differentiable function, and suppose that $f'(a) = 0$.
  - If $f''(a) < 0$, then $f$ has a relative maximum output at $x = a$.
  - If $f''(a) > 0$, then $f$ has a relative minimum output at $x = a$.
  - If $f''(a) = 0$, then no conclusion can be drawn from this test.

The Second Derivative Test is not as powerful as the First Derivative Test, since it can only be applied to critical numbers for $f$ where the second derivative is defined. (The First Derivative Test has no such restrictions.) However, for functions where the second derivative is easy to compute, this test can be a quicker way to classify relative extrema.

**Example 3** *Use the Second Derivative Test to classify the relative extrema for $f(x) = -2x^3+15x^2-24x+1$.*

**Solution.** We must start by identifying the critical numbers for $f$, and this can only be done from the first derivative. Observe that

$$f'(x) = -6x^2 + 30x - 24$$

Now, setting $f'(x) = 0$ tells us

$$\begin{aligned} -6x^2 + 30x - 24 = 0 \quad &\Longrightarrow \quad x^2 - 5x + 4 = 0 \\ &\Longrightarrow \quad (x-1)(x-4) = 0 \\ &\Longrightarrow \quad x = 1 \quad \text{or} \quad x = 4 \end{aligned}$$

Thus, $f$ has two critical numbers, namely $x = 1$ and $x = 4$. Since both critical numbers satisfy $f'(x) = 0$, we may apply the Second Derivative Test to try and classify them. First, observe that

$$f''(x) = -12x + 30$$

Since $f''(1) = -12(1) + 30 > 0$, we know that $f$ has a relative minimum output at $x = 1$. Since $f''(4) = -12(4) + 30 < 0$, we know that $f$ has a relative maximum output at $x = 4$.

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**Problem 5.** Use the Second Derivative Test to classify the relative extrema for $f(x) = 1 + 12x - x^3$.

**HOMEWORK:** Section 4.3 Pages 301-302, Problems 9, 11, 17, 19, 20, 25, 27, 35, 38