If A and B are sets, then we define the *product set* $A \times B$ to be the set of ordered pairs whose first entry is a member of A and whose second entry is a member of B. In symbols, we have

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

Problem 1. What are the members of the set $A \times \emptyset$? Write a conjecture and construct a formal proof.

Problem 2. Construct a formal proof for the following conjecture.

- Suppose A and B are sets. If $A \times B = B \times A$, then $A = B$.

Binary Relation on a Set

A *binary relation* on a set A is any subset of $A \times A$.

1. We say that a binary relation $\mathcal{R}$ on a set A is *reflexive* provided $(x, x) \in \mathcal{R}$ FOR ALL $x \in A$.
2. We say that a binary relation $\mathcal{R}$ on a set A is *symmetric* provided IF $(x, y) \in \mathcal{R}$ THEN $(y, x) \in \mathcal{R}$.
3. We say that a binary relation $\mathcal{R}$ on a set A is *antisymmetric* provided IF $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ THEN $x = y$.
4. We say that a binary relation $\mathcal{R}$ on a set A is *transitive* provided IF $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ THEN $(x, z) \in \mathcal{R}$.

Problem 3. Let A be any nonempty set. Does the *empty* binary relation on A satisfy any of the relational properties listed above? Explain.

Problem 4. Suppose that Bill, Ted, Thelma, Louise, Bing, and Bob each own a house in this order on a straight road. Each house is exactly 0.20 mile apart. Let $A = \{\text{Bill}, \text{Ted}, \text{Thelma}, \text{Louise}, \text{Bing}, \text{Bob}\}$ and consider the binary relation $\mathcal{R}$ on A defined by $(u, v) \in \mathcal{R}$ provided Person u lives no more than one mile from Person v.

Part (a). Is this relation reflexive? Explain.

Part (b). Is this relation symmetric? Explain.

Part (c). Is this relation antisymmetric? Explain.

Part (d). Is this relation transitive? Explain.

> ***Partition of a Set***
>
> Let A be any set. A *partition* of the set A is a collection $\mathbb{P}$ of subsets of A with the property that every member of A is a member of *exactly one* member of $\mathbb{P}$.

Problem 5. Construct a partition of the set $A = \{a, b, c, d, e\}$.

Problem 6. Construct a formal proof of the following conjecture. (Proof by contradiction works well.)

- *Let A be any nonempty set, and suppose* $\mathbb{P}$ *is a partition of A. If* $X, Y \in \mathbb{P}$ *and* $X \cap Y \neq \emptyset$, *then* $X = Y$.

Problem 7. Consider the set $A = \{a, b, c, d, e, f\}$ and the partition defined below.

$$\mathbb{P} = \big\{\{a, e\}, \{b\}, \{c, d, f\}\big\}$$

Part (a). Define a binary relation $\mathcal{R}_{\mathbb{P}}$ on the set A according to the following rule: $(x, y) \in \mathcal{R}_{\mathbb{P}}$ if and only if x and y are members of the same member of $\mathbb{P}$. Construct the binary relation $\mathcal{R}_{\mathbb{P}}$.

Part (b). Is this relation reflexive? Explain.

Part (c). Is this relation symmetric? Explain.

Part (d). Is this relation antisymmetric? Explain.

Part (e). Is this relation transitive? Explain.

Theorem 10.1 *Let A be any set, and suppose $\mathbb{P}$ is a partition of A. If we let*

$$\mathcal{R}_{\mathbb{P}} = \{(x, y) : x, y \in U \ \text{ for some } U \in \mathbb{P}\}$$

then the binary relation $\mathcal{R}_{\mathbb{P}}$ is always reflexive, symmetric, and transitive.

Proof. To prove that $\mathcal{R}_{\mathbb{P}}$ is reflexive, we must show that $(c, c) \in \mathcal{R}_{\mathbb{P}}$ for all $c \in A$. If $c \in A$, then we know there exists $U \in \mathbb{P}$ such that $c \in U$. If we know there exists $U \in \mathbb{P}$ such that $c \in U$, then we know $(c, c) \in \mathcal{R}_{\mathbb{P}}$ by definition. Therefore, if $c \in A$, then we know $(c, c) \in \mathcal{R}_{\mathbb{P}}$.

Problem 8. What argument form was used in the proof that $\mathcal{R}_{\mathbb{P}}$ is reflexive?

Problem 9. Construct a formal proof that $\mathcal{R}_{\mathbb{P}}$ is symmetric.

Problem 10. Construct a formal proof that $\mathcal{R}_{\mathbb{P}}$ is transitive.

Equivalence Relation on a Set   Let A be any set. An *equivalence relation* on the set A is a binary relation $\mathcal{R}$ on A that is reflexive, symmetric, and transitive.

Exercises.

Problem 1. Let A be any set. The *powerset* of A is defined to be the collection of all subsets of A; we will let $\wp(A)$ denote the powerset of A. Define a binary relation SU on $\wp(A)$ according to the rule $(U, V) \in \text{SU}$ if and only if $U \subseteq V$.

Part (a). Give a specific counterexample to show that SU need not be symmetric.

Part (b). Construct a formal proof that SU is reflexive, transitive, and antisymmetric.

Problem 2. Let $\mathbb{N}$ denote the set of positive integers (also known as the *natural numbers*). Define a binary relation D on the set $\mathbb{N}$ according to the rule $(a, b) \in D$ if and only if $b = ka$ for some positive integer k.

Part (a). Give a specific counterexample to show that D need not be symmetric.

Part (b). Construct a formal proof that D is reflexive, transitive, and antisymmetric[1].

Problem 3. Let A be any set. The *diagonal relation* on A is the binary relation $\mathfrak{D}_A$ defined by the rule $(x, y) \in \mathfrak{D}_A$ if and only if $x = y$. Construct a formal proof for the following conjectures.

Part (a). If A is nonempty, then $\mathfrak{D}_A$ is symmetric and antisymmetric.

Part (b). If A is nonempty, then $\mathfrak{D}_A$ is the only nonempty binary relation that is symmetric and antisymmetric. (Proof by contradiction works well here.)

[1] The set D is called the *divisibility relation* on $\mathbb{N}$.

Problem 4. Let A be any set, and suppose that $\mathcal{E}$ is an equivalence relation on A. For each $x \in A$, let

$$E_x = \{y \in A : (x, y) \in \mathcal{E}\}$$

(The sets E_x are called *equivalence classes* for $\mathcal{E}$.) Let $\mathbb{P}_{\mathcal{E}} = \{E_x : x \in A\}$.

Part (a). Suppose that $A = \{a, b, c, d, e, f\}$ and consider the equivalence relation

$$\mathcal{E} = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (a, c), (c, a), (a, b), (b, a), (c, b), (b, c), (e, d), (d, e)\}$$

Construct the equivalence classes for this relation.

Part (b). Construct a formal proof for the following conjecture.

- If A is any set and $\mathcal{E}$ is an equivalence relation on A, then $\mathbb{P}_{\mathcal{E}}$ is a partition of A.

Problem 5. Let $\mathbb{R}$ denote the set of real numbers. Define a binary relation $\mathcal{C}$ on $\mathbb{R}$ according to the rule $(a, b) \in \mathcal{C}$ if and only if there exists some real number r such that $a^2 + b^2 = r^2$.

Part (a). Construct a formal proof that $\mathcal{C}$ is an equivalence relation.

Part (b). What are the equivalence classes for this relation?