In this investigation, we will explore the notion of "function" from a set-based perspective. We begin by expanding our definition of binary relation.

> ***Binary Relation Between Sets***
>
> Let A and B be any sets. A binary relation *from A to B* is any subset of $A \times B$. A binary relation $\mathcal{G}$ from A to B represents the *graph* of a *function* from A to B provided every member of A appears as the first coordinate of exactly one member of $\mathcal{G}$.

Problem 1. A vending machine has seven buttons, labeled 1 through 7. The machine dispenses four different candies, namely Red Hots, Nerds, Twizzlers, and Jolly Ranchers. Twizzlers are the most popular item and can be obtained by punching Button 1, 2, or 3. Nerds can be obtained by punching Button 4 or Button 5. Red Hots and Jolly Ranchers can be obtained by punching Button 6 or Button 7, respectively. Let

$$A = \{\text{Red Hots}, \text{Nerds}, \text{Twizzlers}, \text{Jolly Ranchers}\} \quad B = \{1,2,3,4,5,6,7\}$$

Part (a). Construct the binary relation $\mathcal{G}$ from A to B and the binary relation $\mathcal{H}$ from B to A which represent this situation.

Part (b). Do either of these relations represent the graph of a function? Explain.

Problem 2. Let $\mathbb{Z}$ denote the set of all integers[1], and let $A = \mathbb{Z} \times \mathbb{Z}$. Consider the binary relations $\mathcal{G}$ from A to $\mathbb{Z}$ and the binary relation $\mathcal{H}$ from $\mathbb{Z}$ to A defined by

$$\mathcal{G} = \{((x,y),z) : z = x + y\} \qquad \mathcal{H} = \{(z,(x,y)) : z = x + y\}$$

Do either of these relations represent the graph of a function? Justify your answer.

Let A and B be sets, and suppose $\mathcal{G}$ is the graph of a function from A to B. Since every member of A is paired with exactly one member of B, we can think of the relation $\mathcal{G}$ as defining a "rule" which relates all the members of A to members of B. This "rule" is the *function* represented by $\mathcal{G}$. The members of A are thought of as the *inputs* of this "rule," and the members of B that are paired with members of A are thought of as the *outputs* of this "rule."

[1] The blackboard-bold letter "$\mathbb{Z}$" comes from the German noun "*zahle*" which means "number."

- It is customary to use a separate letter to denote the "rule" represented by $\mathcal{G}$. This letter can be suggestive of the rule, or it can be generic. (The letter f is commonly used.)
- The set A is called the *domain* of the function f, and the set B is called the *codomain* of the function f.
- If $x \in A$, then it is customary to let $f(x)$ denote the second coordinate of the ordered pair in $\mathcal{G}$ whose first coordinate is x. We read this symbol as "the *output* of the function f for the *input* x," or more simply as "f of x."

The notation for representing function output is unfortunate, since we also use juxtaposition of letters to denote real number multiplication. It is therefore very important to indicate clearly when a letter is used to represent a function. This can be done in sentence form by writing something like

"Let f be a function from a set A to a set B."

There is accepted symbolic shorthand for this sentence as well. The expression

"Consider the function $f : A \to B$."

is used to name a function "rule" that assigns the elements of the set A to the set B. Again, care must be taken to avoid confusion with our use of the right arrow in conditional statements.

Preimage of a Codomain Element

Let A and B be any sets and suppose that f is a function from A to B. The *preimage* of any $y \in B$ is defined to be the set

$$\text{Pre}_f(y) = \{x \in A : f(x) = y\}$$

Problem 3. Let $\mathbb{R}$ denote the set of real numbers and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by the output formula

$$f(x) = \sin(\pi x)$$

Part (a). What real numbers constitute the set $\text{Pre}_f(0)$?

Part (b). Are there any real numbers a for which $\text{Pre}_f(a)$ is finite? Explain.

Problem 4. Let A and B be nonempty sets, and suppose f is a function from A to B. Construct a formal proof that the set

$$\mathbb{P}_f = \{\text{Pre}_f(y) : y \in B\}$$

is a partition of the set A.

Problem 5. Let $A = \{1,2,3,4,5,6,7\}$ and let $B = \{a, b, c\}$ and $C = \{w, x, y, z\}$ Consider the partition of A defined by the collection

$$\mathbb{P} = \{\{1,2\}, \{3,6\}, \{4,5,7\}\}$$

Part (a). Is it possible to define a function f from A to B so that $\text{Pre}_f(m)$ is a member of $\mathbb{P}$ for *every* $m \in B$? Give an example or explain why it is not possible.

Part (b). Is it possible to define a function g from A to C so that $\text{Pre}_g(m)$ is a member of $\mathbb{P}$ for *every* $m \in C$? Give an example or explain why it is not possible.

Let A and B be sets, and consider a function $f : A \to B$. If $U \subseteq A$ and $V \subseteq B$, then it is customary to let

$$f(U) = \{f(x) : x \in U\} \qquad \text{Pre}_f(V) = \{x \in A : x \in \text{Pre}_f(y) \text{ for some } y \in V\}$$

Problem 6. Consider the sets A and B defined in Problem 1 above. Let $g : B \to A$ be the function whose graph $\mathcal{H}$ you constructed. Let $U = \{2,5\}$ and construct the set $\text{Pre}_g\big(g(U)\big)$.

Problem 7. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by the formula $f(x) = |x|$. Let $U = \{1,2,-3\}$ and construct the set $\text{Pre}_f\big(f(U)\big)$.

Problem 8. Let X and Y be sets, let $U \subseteq X$ be nonempty, and consider a function $f : X \to Y$.

Part (a). Is it ever possible to have $\text{Pre}_f\big(f(U)\big) = \emptyset$? Explain.

Part (b). Is it ever possible to have $\text{Pre}_f\big(f(U)\big) = U$? Explain.

Problem 9. Let X and Y be sets, let $U \subseteq X$, and consider a function $f : X \to Y$. Based on your work in Problems 6 – 8 , how do you think the sets U and $\text{Pre}_f\big(f(U)\big)$ are related? Write your answer as a conjecture and construct a formal proof.

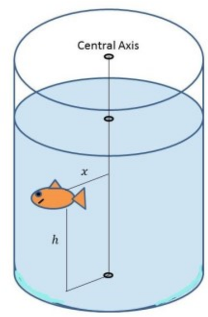
Exercises.

Boris buys a cylindrical glass bowl that has a radius of six inches. He adds water to a depth of nine inches in the bowl and places his goldfish, Natasha, in the bowl. Boris proceeds to watch Natasha nonstop for five minutes as she swims around. Consider the sets

$$T = \{t \in \mathbb{R} : 0 \leq t \leq 5\} \qquad D = \{x \in \mathbb{R} : 0 \leq x \leq 6\} \quad V = \{h \in \mathbb{R} : 0 \leq h \leq 9\}$$

Problem 1. Let's consider the relations $\mathcal{W} \subseteq T \times V$, $\mathcal{X} \subseteq D \times T$, and $\mathcal{Y} \subseteq D \times V$ defined by the following rules.

- The pair $(t, h) \in \mathcal{W}$ and $(h, t) \in \mathcal{X}$ if and only if Natasha was at height h (measured in inches from the bottom of the bowl) at time value t during the five minutes Boris watched her.
- The pair $(x, h) \in \mathcal{Y}$ if and only if Natasha was at height h and horizontal distance x (measured in inches from the central axis of the bowl) at least once in the five minutes Boris watched her.



Part (a). Does $\mathcal{W}$ represent the graph of a function from T to V? Justify your answer.

Part (b). Does $\mathcal{X}$ represent the graph of a function from D to T? Justify your answer.

Part (c). Does $\mathcal{Y}$ represent the graph of a function from D to V? Justify your answer.

Problem 2. Let A and B be sets, and consider a function $f : A \to B$. Construct a formal proof for the following conjecture.

- *If* $U \subseteq V$ *and* $V \subseteq A$, *then* $f(U) \subseteq f(V)$.

Problem 3. Let A and B be sets, and consider a function $f : A \to B$. Construct a formal proof for the following conjectures.

- *If* $U \subseteq A$ *and* $V \subseteq A$, *then* $f(U \cap V) \subseteq f(U) \cap f(V)$.
- *If* $U \subseteq A$ *and* $V \subseteq A$, *then* $f(U \cup V) = f(U) \cup f(V)$.

Problem 4. Let A and B be sets, and consider a function $f : A \to B$. Construct a formal proof for the following conjecture.

- *If* $U \subseteq B$ *and* $V \subseteq B$, *then* $\text{Pre}_f(U \cap V) = \text{Pre}_f(U) \cap \text{Pre}_f(V)$.

Problem 5. Let A and B be sets, and consider a function $f : A \to B$. Construct a formal proof for the following conjecture.

- *If* $U \subseteq B$, *then* $\text{Pre}_f(B - U) = A - \text{Pre}_f(U)$.