The definition of a function requires that every member of the domain be paired with exactly one member of the codomain. However, the definition tells us nothing about how members of the codomain are paired with members of the domain. We can classify functions by describing the possibilities.

> ***Three Important Types of Functions***
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> Let A and B be any sets and suppose that f is a function from A to B.
> - We say that f is a *surjection* provided $\text{Pre}_f(y)$ is nonempty for all $y \in B$.
> - We say that f is an *injection* provided $\text{Pre}_f(y)$ contains at most one member for all $y \in B$.
> - We say that f is a *bijection* provided $\text{Pre}_f(y)$ contains exactly one member for all $y \in B$.

Problem 1. Let $2\mathbb{Z}$ denote the set of even integers, and consider the function $f : \mathbb{Z} \to \mathbb{Z}$ and the function $g : \mathbb{Z} \to 2\mathbb{Z}$ defined by the formulas

$$f(n) = 2n \qquad g(n) = 2n$$

Is either of these functions a surjection? Justify your answer.

Problem 2. Let $\mathbb{Z}^+$ denote the set of positive integers, and consider the function $f : \mathbb{Z} \to \mathbb{Z}$ and the function $g : \mathbb{Z}^+ \to \mathbb{Z}$ defined by the formulas

$$f(n) = n^2 \qquad g(n) = n^2$$

Is either of these functions an injection? Justify your answer.

Let A and B be sets, and suppose $f : A \to B$ is a function. The *range* of f is defined to be the set

$$\text{Ran}(f) = \{y \in B : \text{Pre}_f(y) \text{ is nonempty}\}$$

Note that f is a surjection if and only if $B = \text{Ran}(f)$.

Theorem 12.1 *Let a and b be fixed real numbers with $a < b$ and let $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$. There exists a bijection $f : [0,1) \to [a, b)$.*

Proof. We will demonstrate the function defined by $f(x) = bx + a$ serves as the desired bijection.

Problem 3. Suppose that $y \in [a, b)$ and use algebra to determine a real number $x \in [0,1)$ such that $f(x) = y$. What does this computation prove?

Problem 4. Suppose that $u, v \in [0,1)$ are such that $f(u) = f(v)$. Use algebra to show that $u = v$. What does this computation prove?

Problem 5. Let $\mathbb{R}^0$ denote the set of nonnegative real numbers. Show that the function $f : [0,1) \to \mathbb{R}^0$ defined by the formula

$$f(x) = \frac{x}{1 - x}$$

is a bijection.

> ***Composing Functions***
>
> Let A, B, and C be sets, and consider functions $f : A \to B$ and $g : B \to C$. The *composite* function $g \circ f$ from A to C is defined according to the rule $[g \circ f](x) = g\big(f(x)\big)$ for all $x \in A$.

Problem 6. Let A, B, and C be sets, and consider functions $f : A \to B$ and $g : B \to C$. Construct a formal proof of the following conjecture.

- *If f and g are surjections, then $g \circ f$ is also a surjection.*

Let X be any set. The *identity function* on the set X is the function $I_X : X \to X$ defined by $I_X(x) = x$.

Problem 7. Let A and B be sets, and suppose $f : A \to B$ is a function.

Part (a). Suppose that f is a surjection. For all $y \in B$, we know $\text{Pre}_f(y)$ is nonempty. Define a new function $g : B \to A$ by choosing any $x \in \text{Pre}_f(y)$ and letting $g(y) = x$. Prove that $f \circ g = I_B$.

Part (b). On the other hand, suppose there exists a function $h : B \to A$ such that $f \circ h = I_B$. Prove that f must be a surjection.

In Problem 7, you proved two conjectures, namely for any function $f : A \to B$,

- *If f is a surjection, then there exists a function $g : B \to A$ such that $f \circ g = I_B$.*
- *If there exists a function $g : B \to A$ such that $f \circ g = I_B$, then f is a surjection.*

Notice that these sentences represent conditional statements that are *converses* of each other. By proving that the conditional statement and its converse are true, you established the truth of a *biconditional* statement, namely the statement represented by the sentence

A function $f : A \to B$ is a surjection if and only if there exists a function $g : B \to A$ such that $f \circ g = I_B$.

This function g is known as a *right inverse* for the function f.

Exercises.

Problem 1. Let A and B be sets, and suppose $f : A \to B$ is an injection. Explain why $f : A \to \text{Ran}(f)$ is a bijection. (We are simply ignoring the members of the codomain which have no preimage.)

Problem 2. Let A, B, and C be sets, and consider functions $f : A \to B$ and $g : B \to C$. Construct a formal proof of the following conjecture.

- *If f and g are injections, then $g \circ f$ is also an injection.*

Problem 3. Let A, B, and C be sets, and consider functions $f : A \to B$ and $g : B \to C$. Is the following conjecture a theorem? Explain.

- *If f and g are bijections, then $g \circ f$ is also a bijection.*

Problem 4. Let A, B, and C be sets, and consider functions $f : A \to B$ and $g : B \to C$. Construct a formal proof of the following conjectures.

- *If $g \circ f$ is a surjection, then g is also a surjection.*
- *If $g \circ f$ is an injection, then f is also an injection.*

Problem 5. Construct a formal proof of the following conjecture.

- *A function $f : A \to B$ is an injection if and only if there exists a function $g : B \to A$ such that $g \circ f = I_A$.*

Hint: If f is an injection, does it matter how you define the output of g on the set $B - \text{Ran}(f)$? (This function g is called a *left inverse* for the function f.)

Problem 6. Consider the function $f : \mathbb{R} \to \mathbb{R}^0$ defined by the rule $f(x) = x^2$. Construct two different right inverses for the function f.

Problem 7. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by the rule $f(x) = 2x$. Construct two different left inverses for the function f. (Consider piecewise-defined functions.)

Problem 8. Let A and B be sets, and suppose $f : A \to B$ is a function. Construct a formal proof for the following conjecture.

- If f has a left inverse g and a right inverse h, then $g(y) = h(y)$ for all $y \in B$.

Problem 7 tells us that, if a function $f : A \to B$ is a bijection, then there exists exactly one function $g : B \to A$ such that $f\big(g(y)\big) = y$ for all $y \in B$ and $g\big(f(x)\big) = x$ for all $x \in A$. This special function is known as the *inverse* for the function f. It is common (but terrible) practice to let f^{-1} denote the inverse of a function f when it exists.

Problem 9. Let A and B be sets, and suppose $f : A \to B$ is a bijection. Explain why its inverse function $g : B \to A$ is also a bijection.

Problem 10. Suppose a and b are fixed real numbers such that $a < b$. Explain why there is a bijection from the set $\mathbb{R}^0$ to the set $[a, b)$.

Problem 11. Consider the function $f : (0,1) \to \mathbb{R}$ defined according to the rule

$$f(x) = \begin{cases} \dfrac{x}{1/2 - x} & if \quad 0 < x < 1/2 \\ \dfrac{x - 1/2}{1 - x} & if \quad 1/2 \le x < 1 \end{cases}$$

Prove that this function is a bijection.

Problem 12. Consider the function $f : [0,1) \to (0,1)$ defined according to the rule[1]

$$f(x) = \begin{cases} 2^{-1} & \text{if} \quad x = 0 \\ 2^{-(n+1)} & \text{if} \quad x = 2^{-n} \\ x & \text{otherwise} \end{cases}$$

Prove that this function is a bijection.

[1] Twink (https://math.stackexchange.com/users/91974/twink), Bijection between open and closed interval, URL (version: 2015-03-22): https://math.stackexchange.com/q/1201347