In this investigation, we will introduce a very useful extension of the notion of "finite" set. We begin with a definition.

Countable Sets

We say that a set A is *countable* provided A is finite or there exists a bijection $f : \mathbb{Z}^+ \to A$.

Problem 1. Prove that the set $\mathbb{Z}$ is countable by constructing a bijection $f : \mathbb{Z}^+ \to \mathbb{Z}$.

If a set A is infinite and countable, then there exists a bijection $f : \mathbb{Z}^+ \to A$. It is customary to let $a_i = f(i)$; the representation $A = \{a_i : i \in \mathbb{Z}^+\}$ is called a *listing* or a *denumeration* of A. In this context, the members of $\mathbb{Z}^+$ are known as the *indices* (plural of *index*) for the set A.

Problem 2. Suppose that A and B are countable infinite sets. Let $A = \{a_i : i \in \mathbb{Z}^+\}$ and $B = \{b_j : j \in \mathbb{Z}^+\}$ be listings for these sets. Construct a formal proof that $A \cup B$ is countable. (Think about your solution to Problem 1.)

Problem 5 can be extended to the union of any *finite* family of countable sets --- if $A_1, \ldots, A_n$ are all countable sets, then $A_1 \cup \ldots \cup A_n$ is also countable. We could prove this fact by mimicking the approach to Problem 5, but there is a much more convenient approach based on an assumption we make about the positive integers.

> ***Axiom of Induction***
>
> Suppose $P \subseteq \mathbb{Z}^+$ has the following properties:
> a. We have $1 \in P$.
> b. If $x \in P$, then $x + 1 \in P$.
>
> Under these conditions, we assume $P = \mathbb{Z}^+$.

The Axiom of Induction provides us with a new proof technique known as the *Method of Induction*. Here is how the Method of Induction works.

Suppose we have a countable infinite family of statements $\mathcal{S} = \{s_i : i \in \mathbb{Z}^+\}$, and suppose we wish to establish that every member of $\mathcal{S}$ is true. Let P represent the set of all $i \in \mathbb{Z}^+$ such that s_i is true.

1. We first construct a formal proof that s_1 is true (thus telling us $1 \in P$).
2. Next, we *assume* s_i is true (that is, assume $i \in P$) and use this assumption to prove that s_{i+1} is true (so that we may conclude $i + 1 \in P$). (This is known as the *inductive step* of the proof.)

Once we have accomplished these two steps, we invoke the Axiom of Induction to conclude that $P = \mathbb{Z}^+$; in other words, we conclude that every member of $\mathcal{S}$ is true.

Theorem 13.1 *If $A_1, \ldots, A_n$ are all countable sets, then $A_1 \cup \ldots \cup A_n$ is also countable.*

Proof. We will use the Method of Induction. For each positive integer n, let s_n be the statement represented by the sentence

- *If $A_1, \ldots, A_n$ are all countable sets, then $A_1 \cup \ldots \cup A_n$ is also countable.*

Let $P = \{n \in \mathbb{Z}^+ : s_n \text{ is true}\}$. We want to prove that $P = \mathbb{Z}^+$. Note that Statement s_1 is represented by the sentence

- *If A_1 is a countable set, then A_1 is a countable set.*

This is obviously a true statement; hence, we know that $1 \in P$. Now, let $j \in P$; in other words, assume Statement s_j is true. We want to prove this assumption allows us to conclude that $j + 1$ is also a member of P.

Assume True: *If $A_1, \ldots, A_j$ are all countable sets, then $A_1 \cup \ldots \cup A_j$ is also countable.*
Prove True: *If $A_1, \ldots, A_{j+1}$ are all countable sets, then $A_1 \cup \ldots \cup A_{j+1}$ is also countable.*

Now, observe $A_1 \cup \ldots \cup A_{j+1} = (A_1 \cup \ldots \cup A_j) \cup A_{j+1}$. We have *assumed* $B = A_1 \cup \ldots \cup A_j$ is countable; hence, Problem 5 tells us we must conclude $B \cup A_{j+1}$ is also countable. In other words, we must conclude that s_{j+1} is a true statement. The Axiom of Induction therefore allows us to conclude that $P = \mathbb{Z}^+$ as desired.

Note that Theorem 13.1 allows for the possibility that some (or all) of the countable sets are *finite*.

When we apply the Method of Induction, we have constructed a *proof by induction*. Proof by induction is not by itself a valid argument form like modus ponens, modus tollens, or the hypothetical syllogism. Rather, it is a strategy that employs valid argument forms to establish the hypotheses of an axiom. For this reason, the Method of Induction can only be applied in contexts where the Axiom of Induction makes sense.

Problem 3. Use the Method of Induction to prove the following conjecture.

- *If* $S = \{s_1, \dots, s_n\}$ *is a finite set, then* $\wp(S)$ *contains exactly* 2^n *members.*

Problem 4. Use the Method of Induction to establish the following conjecture.

- *If* n *is any positive integer, then*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Exercises.

Problem 1. Let a and b be integers with $a \neq 0$ and consider the function $f : \mathbb{Z}^+ \to \mathbb{Z}$ defined by the formula $f(x) = ax + b$. Prove that $\operatorname{Ran}(f)$ is countable.

Problem 2. Suppose $T \subseteq \mathbb{Z}^+$ has no smallest member. For each positive integer n, let s_n be the statement represented by the sentence

- *No positive integer less than or equal to* n *is a member of* T.

Let $P = \{n \in \mathbb{Z}^+ : s_n \text{ is true}\}$. Use the Method of Induction to prove $T = \emptyset$.

Problem 2 provides proof of the following conjecture: *If* T *is any nonempty subset of* $\mathbb{Z}^+$*, then* T *has a smallest member*. This fact is called the *Well-Ordering Property* for the positive integers.

Problem 3. Suppose $T \subseteq \mathbb{Z}^+$ is infinite. Construct a sequence $f : \mathbb{Z}^+ \to T$ according to the following rule.

- Let $f(1)$ be the smallest member of the set T.
- If $n > 1$, let $f(n)$ be the smallest member of the set $T - \{f(1), \dots, f(n-1)\}$.

Part (a). For each positive integer n, let s_n be the statement represented by the sentence

- *The integer* n *is assigned a member of* T *by the rule* f.

Let $P = \{n \in \mathbb{Z}^+ : s_n \text{ is true}\}$ and use the Method of Induction to prove that $P = \mathbb{Z}^+$. (This tells us that the rule for f does indeed assign every member of $\mathbb{Z}^+$ to some member of T.)

Part (b). For each positive integer n, let t_n be the statement represented by the sentence

- *We have* $j \leq f(j)$ *for all* $1 \leq j \leq n$.

Let $Q = \{n \in \mathbb{Z}^+ : t_n \text{ is true}\}$ and use the Method of Induction to prove that $Q = \mathbb{Z}^+$. (Proof by contradiction works well for the inductive step.)

Problem 4. Suppose $T \subseteq \mathbb{Z}^+$ is infinite and consider the sequence $f : \mathbb{Z}^+ \to T$ defined in Problem 4.

Part (a). Suppose that m and n are positive integers such that $m < n$. Explain why it is impossible to have $f(m) = f(n)$.

Part (b). Let $a \in T$ and suppose there exist $n \in \mathbb{Z}^+$ such that $a \neq f(j)$ for $1 \leq j \leq n$. Explain why we we must have $f(n) < a$.

Part (c). Suppose $a \in T$ is such that $a \neq f(n)$ for *all* $n \in \mathbb{Z}^+$. Part (b) tells us $f(n) < a$ for all positive integers n. Use Part (b) of Problem 4 to explain why this is impossible.

Part (d). Use Parts (a) and (c) to prove that f is a bijection.

Taking Problems 3 and 4 together, we have proved the following conjecture.

- *If T is any subset of $\mathbb{Z}^+$, then T is countable.*

Problem 5. Use the previous result and Homework Problem 1 of Investigation 12 to prove the following conjecture.

- *If A is any set for which an injection $f : A \to \mathbb{Z}^+$ exists, then A is countable.*

Problem 6. Suppose X is a countable infinite set, and let $g : X \to \mathbb{Z}^+$ be a bijection. Prove the following conjecture.

- *If A is any nonempty subset of X, then A is countable.*

Hint: Use the function g to construct an injection $f : A \to \mathbb{Z}^+$ and then use Problem 5.

Problem 7. The *Fundamental Theorem of Arithmetic* tells us that every positive integer larger than 1 has exactly one representation as a product of prime integers (up to the order of the prime factors). Define a function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ according to the rule

$$f(x, y) = 2^x \cdot 3^y$$

Prove that the function f is an injection. (Hence, the set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable by Problem 6.)

Problem 8. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. That is, let

$$\mathbb{Q}^+ = \left\{\frac{m}{n} : m, n \in \mathbb{Z}^+\right\}$$

Use Problems 6 and 7 to prove that $\mathbb{Q}^+$ is countable.

Problem 9. Use Theorem 13.1 and Problem 8 to prove that the set $\mathbb{Q}$ of *all* rational numbers is countable. (Think about the set $\mathbb{Q}^-$ of *negative* rational numbers.)