In the late 1800's, Georg Cantor developed a means for comparing sets that has become one of the pillars of modern set theory. We will introduce his approach in this investigation.

Comparing Sets

Let A and B be any sets.

- We say that A is *less numerous* than B provided there exists an injection $f : A \to B$ but no surjection.
- We say that A is is *more numerous* than B provided there exists a surjection $f : A \to B$ but no injection.
- We say that A and B are *equinumerous* provided there exists a bijection $f : A \to B$.

Problem 1. Cantor's idea for comparing sets grew out of the way we compare finite sets. Consider the sets $A = \{a, b, c, d, e\}$ and $B = \{w, x, y\}$. Use Cantor's approach to prove that A is more numerous than B.

Problem 2. Let A be any infinite subset. Construct an injection $f : \mathbb{Z}^+ \to A$.

In light of the previous two investigations, we know that many familiar sets of numbers are equinumerous. For example, in Investigation 12, we proved that all sets of the form $[a, b)$ are equinumerous, where a and b are real numbers with $a < b$. In the exercises for that investigation, you also showed that the set $(0,1)$ and the set $\mathbb{R}$ of real numbers are equinumerous. In Investigation 13, we proved that if T is a countable infinite set and A is any infinite subset of T, then A and T are equinumerous. In fact, we proved the following result.

- *There is no infinite set less numerous than the set* $\mathbb{Z}^+$.

There exist sets that are *not* countable, as the following result demonstrates.

Theorem 14.1 *If A is any set, then A is less numerous than its powerset.*

Proof. Recall that the powerset of the set A is defined to be the set $\wp(A)$ of all subsets of A. The function $I : A \to \wp(A)$ defined by $I(x) = \{x\}$ is clearly an injection. We must prove that there is no surjection from A to $\wp(A)$.

Problem 3. If A is empty, explain why A is less numerous than $\wp(A)$.

Problem 4. Suppose A is nonempty, and suppose that $f : A \to \wp(A)$ is any function. For each $u \in A$, there exists $V_u \in \wp(A)$ such that $f(u) = V_u$. Consider the *set*

$$X = \{u \in A : u \notin V_u\}$$

Prove that it is not possible for there to exist $b \in A$ such that $f(b) = X$. (Hint: Suppose such b exists. Either $b \in X$ or $b \notin X$; consider both possibilities.)

Theorem 14.2 *The set of real numbers is not countable.*

Proof. We already know that the set of real numbers and the set $[0,1)$ are equinumerous, so it will suffice to prove that $[0,1)$ is not countable. To this end, let $f : \mathbb{Z}^+ \to [0,1)$ be any function. We will prove that f is not a surjection. For each positive integer j, consider the output $f(j)$. We know there exists a function $d_j : \mathbb{Z}^+ \to \mathbb{Z}^+ \cup \{0\}$ such that

$$f(j) = \sum_{n=1}^{\infty} d_j(n) \cdot 10^{-n}$$

(This is a so-called *decimal representation* for the real number $f(j)$.) Now, define a new function $d_y : \mathbb{Z}^+ \to \mathbb{Z}^+ \cup \{0\}$ according to the rule

$$d_y(n) = \begin{cases} 2 & \text{if} \quad d_n(n) \in \{0,1\} \\ d_n(n) - 1 & \text{otherwise} \end{cases}$$

Problem 5. Consider a function $f : \mathbb{Z}^+ \to [0,1)$ whose first six outputs are shown below.

$f(1) = 0.101201000\ldots \quad f(2) = 0.113421100\ldots \quad f(3) = 0.3214513000\ldots$
$f(4) = 0.4281257000\ldots \quad f(5) = 0.49802187000\ldots \quad f(6) = 0.3419518000\ldots$

What are the first six outputs of the function d_y?

Problem 6. Consider a function $f : \mathbb{Z}^+ \to [0,1)$ whose first six outputs are shown below.

$$f(1) = 0.396781000 \ldots \quad f(2) = 0.9954821000 \ldots \quad f(3) = 0.7103517000 \ldots$$
$$f(4) = 0.2104572000 \ldots \quad f(5) = 0.31870451000 \ldots \quad f(6) = 0.88769912000 \ldots$$

What are the first six outputs of the function d_y?

Problem 7. Consider the real number y defined by the decimal representation

$$y = \sum_{n=1}^{\infty} d_y(n) \cdot 10^{-n}$$

Explain why it is not possible to have $y = f(j)$ for some positive integer j.

We will conclude this investigation by proving one of the most important results from the era of set theory development.

Theorem 14.3 (Schroder-Bernstein Theorem)
Suppose that A and B are sets. If there exists an injection $f : A \to B$ and an injection $g : B \to A$, then A and B are equinumerous.

Proof. If $\text{Ran}(f) = B$, then f is a bijection; and there is nothing to show. Suppose this is not the case, and let $B_0 = B - \text{Ran}(f)$. For each positive integer n, let $[f \circ g]^n$ represent the composition of the function $f \circ g$ with itself n times.

Problem 8. For each positive integer n, let $B_n = [f \circ g]^n(B_0)$. Use the Method of Induction to prove that $B_{n+1} = [f \circ g](B_n)$ for every positive integer n.

Problem 9. Suppose $y \in B$ is such that $y \notin B_j$ for all $j \in \mathbb{Z}^+$. Prove that $y \in \text{Ran}(f)$.

Since f is an injection, we know $\text{Pre}_f(y)$ contains a single member for each $y \in \text{Ran}(f)$. Let x_y represent this member, and define a function $h : B \to A$ according to the rule

$$h(y) = \begin{cases} g(y) & \text{if} \quad y \in B_n \text{ for some nonnegative integer } n \\ x_y & \quad \text{otherwise} \end{cases}$$

We will prove that h is a bijection. To begin, suppose $u, v \in B$ are such that $h(u) = h(v)$. There are two possibilities for u --- either $u \in B_n$ for some n or $u \notin B_j$ for all $j \in \mathbb{Z}^+$.

Problem 10. Suppose first that $u \in B_n$ for some n. Either $v \in B_k$ for some positive integer k, or $v \notin B_j$ for all $j \in \mathbb{Z}^+$. Since $u \in B_n$, we know $u = [f \circ g]^n(x)$ for some $x \in B_0$.

Part (a). Suppose $v \in B_k$ for some positive integer k and use the definition of h to explain why we must conclude $u = v$.

Part (b). Suppose instead that $v \notin B_j$ for all $j \in \mathbb{Z}^+$. It follows that $h(v) = x_v$; therefore, we know $v = f\big(h(v)\big) = f\big(h(u)\big)$. Prove that this is impossible.

Problem 11. On the other hand, suppose that $u \notin B_j$ for all $j \in \mathbb{Z}^+$. In light of Problem 10 (b), we must assume $v \notin B_j$ for all $j \in \mathbb{Z}^+$ as well. Use the definition of h to explain why we must conclude $u = v$.

Problems 10 and 11 tell us that the function h is an injection. It remains to prove that h is a surjection. To this end, let $x \in A$. We need to show that $\text{Pre}_h(x)$ is nonempty. Consider $f(x)$.

Problem 12. Either $f(x) \in B_n$ for some positive integer n, or $f(x) \notin B_n$ for all $n \in \mathbb{Z}^+$.

Part (a). First, suppose $f(x) \notin B_n$ for all $n \in \mathbb{Z}^+$. Explain why this implies $f(x) \in \text{Pre}_h(x)$.

Part (b). Suppose $f(x) \in B_n$ for some positive integer n. This tells us that $f(x) \in [f \circ g](B_{n-1})$ by Problem 8 above. Why can we conclude there exist $y \in B_{n-1}$ such that $x = g(y)$?

Part (c). Explain why we must conclude $y \in \text{Pre}_h(x)$.

Let's examine a concrete example of the Schroeder-Bernstein Theorem in action. Consider the sets $\mathbb{Z}^+$ and $2\mathbb{Z}^+$ along with the injections $f : \mathbb{Z}^+ \to 2\mathbb{Z}^+$ and $g : 2\mathbb{Z}^+ \to \mathbb{Z}^+$ defined by the rules

$$f(x) = 4x \qquad g(y) = 3y$$

Note that $B_0 = \{4k + 2 : k \in \mathbb{Z}^+\}$. This is the subset of $2\mathbb{Z}^+$ that was "missed" by the function f. The goal of the construction in the proof of the Schroeder-Bernstein Theorem is to "spread out" the image of f and "redistribute" B_0 in the gaps.

Note that $g(B_0) = \{12k + 6 : k \in \mathbb{Z}^+\}$. The action of g has "sprinkled" the set B_0 throughout the set $\mathbb{Z}^+$. Now, observe

$$\begin{gathered} B_1 = [f \circ g](B_0) = \{48k + 24 : k \in \mathbb{Z}^+\} \\ B_2 = [f \circ g](B_0) = \{576k + 288 : k \in \mathbb{Z}^+\} \\ \vdots \\ B_n = [f \circ g](B_{n-1}) = \{12^n(4k + 2) : k \in \mathbb{Z}^+\} \end{gathered}$$

Defining the function h according to the rule provided in the proof, we obtain the formula

$$h(y) = \begin{cases} 3y & \text{if} \quad y = 12^n(4k+2) \text{ for some } n \in \mathbb{Z}^+ \cup \{0\}, k \in \mathbb{Z}^+ \\ y/4 & \text{otherwise} \end{cases}$$

Exercises.

Problem 1. Let A and B be sets, and suppose that A is less numerous than B.

Part (a). Let $f : A \to B$ be an injection. Use this function to construct a surjection $g : B \to A$.

Part (b). Explain why the Schroeder-Bernstein Theorem now implies B is more numerous than A.

Problem 2. Suppose that B is a set that is not countable. Construct an injection $h : \mathbb{Z}^+ \to B$.

Problem 3. Let A be a countable set, and let B be a set that is not countable. The set A may be finite or infinite; regardless, there exists an injection $g : A \to \mathbb{Z}^+$. Consider the set $A \cup B$.

Part (a). Construct an injection $f : B \to A \cup B$.

Part (b). Let $h : \mathbb{Z}^+ \to B$ be any injection. Let $C = B - h(\mathbb{Z}^+)$, and consider $F : A \cup B \to B$ defined by the formula

$$F(x) = \begin{cases} x & \text{if} \quad x \in C \\ h(2n) & \text{if} \quad x = h(n) \\ h(2g(x) + 1) & \text{if} \quad x \in A \end{cases}$$

Prove that F is an injection.

Problem 3 and the Schroeder-Bernstein Theorem together tell us that the following conjecture is true.

- *If A is a countable set and B is a set that is not countable, then $A \cup B$ is a set that is not countable.*

Problem 4. Suppose that $\mathcal{U} = \{U_i : i \in \mathbb{Z}^+\}$ is a countable infinite family of sets, and suppose that each U_i is also countable. Let $C = \{a : a \in U_i \text{ for some } i \in \mathbb{Z}^+\}$. The set C is called the *union* of the family $\mathcal{U}$. For each $U_i \in \mathcal{U}$, let $f_i : U_i \to \mathbb{Z}^+$ be an injection.

Part (a). Let $a \in C$. Explain why there exists a *smallest* index i_a such that $a \in U_{i_a}$.

Part (b). Define a function $g : A \to \mathbb{Z}^+ \times \mathbb{Z}^+$ according to the rule $g(a) = \left(i_a, f_{i_a}(a)\right)$. Prove this function is an injection. (Hence, A is countable.)