In this investigation, we will explore a compound logical statement that is quite different from the ones introduced in the previous investigation. This family of statements has profound importance to reasoning.

When two ascertainable sentences are combined using the constructions

"IF [Sentence 1] THEN [Sentence 2]."
"[Sentence 1] IMPLIES [Sentence 2]."

the new sentence is said to be *conditional*. Sentence 1 in a conditional sentence is called the *antecedent* of the conditional, and Sentence 2 is called the *consequent* of the conditional[1]. We understand conditional sentences to assert that the antecendent somehow *implies*[2] the consequent.

Problem 1. Consider the following conditional sentences.

1. "*If Adolf Hitler is a military genius, then I have wings to fly.*"
2. "*If I am in an automobile, then I am in a horseless carriage.*"
3. "*If Bob poured nitric acid down the sink, then brown gas will come out of the drain.*"
4. "*If all men will go bald and I am a man, then I will go bald.*"

How does the notion of "implies" differ in each of these sentences? Explain.

We would like for conditional sentences to be ascertainable; however, for this to happen, we need a way of discerning the truth-value of the logical statement it represents. Of course, the simplest way to do this would be to connect the truth-value of the statement directly to the truth-values of the antecedent and consequent (thereby making the underlying a truth-functional statement).

Problem 2. The exact nature of "implies" is different in each of the examples above, but there is an assumption about the relationship between the truth-values of the antecendent and consequent that is common to all of them. What would you say this assumption is?

[1] The word "antecedent" is French and means "comes before;" the word "consequent" is also French and means "comes after." Their use in logic dates to the mid 1500's.
[2] The word "imply" comes from the Lain verb "*implicare*" which mean "to involve."

> ***Conditional Statement***
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> Let p and q denote logical statements. The *conditional statement* is defined to be the compound statement $\sim(p \wedge \sim q)$. This statement is usually denoted by the symbol $p \to q$.

Problem 3. Let p and q denote logical statements and construct the truth table for the conditional statement $p \to q$.

Logicians say the conditional statement asserts that p *materially implies* q. Material implication does not guarantee any "connection" between the statements p and q beyond that stated in the definition.

It is common practice to express an ascertainable sentence representing a conditional statement $p \to q$ using the structure

"If [Sentence representing p] THEN [Sentence representing q]."

Of course, this is problematic, since we have already seen that "*If … Then*" sentences represent many kinds of implication in our language. It is therefore important to keep in mind that, when we refer to an "*If … Then*" sentence as being *ascertainable*, we are considering material implication only.

To see why this caveat is important, consider the following conditional sentences.

- "*If oxygen molecules absorb longer wavelengths of light, then the sky is blue.*"
- "*If Tommy's tummy is tubby, then the sky is blue.*"

A chemist would likely feel quite comfortable asserting that the first sentence represents a true implication. However, this chemist would probably feel much less comfortable asserting the same for the second sentence. The first sentence, like Example 3 from Problem 1, presents a *causal* connection between antecedent and consequent--- one which can be tested by experimentation. The second sentence offers no such sense of "connection".

Nevertheless, if we assume "*The sky is blue*" is a true statement and assume the sentences to be ascertainable, then a logician would accept *both* underlying conditional statements as true. This is because logicians discern the truth value of conditional statements via *material* implication --- the implication is TRUE precisely when the antecedent is FALSE or the consequent is TRUE.

Problem 4. Does the ascertainable sentence below represent a true statement? Justify your thinking.

- "*If Mother Theresa was a man, then the moon orbits the Earth.*"

Paradox of Material Implication

A logical statement that is TRUE is materially implied by *every* logical statement, and a logical statement that is FALSE materially implies *every* logical statement.

Problem 5. Let p and q denote logical statements. Construct the truth table for the compound statement $(p \to q) \wedge (q \to p)$.

The logical statement appearing in Problem 5 is called the *biconditional* statement. The biconditional statement is denoted by the symbol $p \leftrightarrow q$ and is read "p if and only if q."

Problem 6. Let p and q denote logical statements. Complete the following truth tables.

p	q	$q \to p$	$p \to (q \to p)$
TRUE	TRUE		
FALSE	TRUE		
TRUE	FALSE		
FALSE	FALSE		

p	q	$\sim p$	$p \to q$	$\sim p \to (p \to q)$
TRUE	TRUE			
FALSE	TRUE			
TRUE	FALSE			
FALSE	FALSE			

A *tautology*[3] is a logical statement that is always TRUE. For example, the logical statement $p \vee \sim p$ is a tautology. In Problem 6 you showed that the logical statements $p \to (q \to p)$ and $\sim p \to (p \to q)$ are both tautologies.

Problem 7. Explain how the tautologies $p \to (q \to p)$ and $\sim p \to (p \to q)$ express the Paradox of Material Implication in symbolic form.

[3] The word "tautology" comes from the Latin noun "*tautologia*" which means "representation of the same thing using different words."

Exercises

Problem 1. Let p and q denote the antecedent and consequent statements in the conditional statement represented by the ascertainable sentence below.

- "*If you give me a hamburger today, then I will gladly pay you Tuesday.*"

Part (a). Construct the ascertainable sentence representing the conditional statement $q \to p$. Construct the truth table for $q \to p$.

Part (b). Construct the ascertainable sentence representing the conditional statement $\sim q \to \sim p$. Construct the truth table for $\sim q \to \sim p$.

Part (c). Construct the ascertainable sentence representing the conditional statement $\sim p \to \sim q$. Construct the truth table for $\sim p \to \sim q$.

Variations on the Conditional Statement

Let p and q be logical statements.

- The conditional statement $q \to p$ is called the *converse* of the statement $p \to q$.
- The conditional statement $\sim q \to \sim p$ is called the *contrapositive* of the statement $p \to q$.
- The conditional statement $\sim p \to \sim q$ is called the *inverse* of the statement $p \to q$.

Problem 2. Based on your truth tables in Problem 1, which of the variations on the conditional statement $p \to q$ are materially equivalent?

Problem 3. Do any of the variations on the conditional statement $p \to q$ denote the *negation* of the statement $p \to q$? Justify your answer.

Problem 4. Let p and q denote logical statements. Is the statement $(p \wedge q) \vee (\sim p \wedge \sim q)$ materially equivalent to the biconditional statement $p \leftrightarrow q$? Use a truth table to justify your answer.

Problem 5. Let p and q denote logical statements. The expression $\sim(p \wedge \sim q)$ is sometimes called the *conjunctive* form of the conditional statement. In the exercise set for Investigation 2, you explored some of the rules for manipulating symbolic expressions. Use these rules to derive the *disjunctive* form of the conditional statement, namely $p \vee \sim q$. Show each step in your derivation and state which rule you used.

Problem 6. Let $\mathbb{P}$ and $\mathbb{Q}$ be classes. Modern logicians consider the following sentences to represent materially equivalent statements.

- *All members of the class $\mathbb{P}$ are members of the class $\mathbb{Q}$.*
- *If x is a member of the class $\mathbb{P}$, then x is a member of the class $\mathbb{Q}$.*

Explain why this assumption is consistent with the truth table defining the conditional statement.

Problem 7. Let $\mathbb{P}$ and $\mathbb{Q}$ be classes. Construct the conditional statement that is materially equivalent to the statement represented by the sentence below.

- *No member of the class $\mathbb{P}$ is a member of the class $\mathbb{Q}$.*

Problem 8. Is every conditional statement materially equivalent to a categorical statement? Justify your answer.

Problem 9. Use the conjunctive or disjunctive form of the conditional statement to construct conjunctions or disjunctions that are materially equivalent to existential affirmative and existential negative statements.

Problem 10. Let p and q denote logical statements. Use truth tables to determine if the following statements are tautologies.

$$([p \rightarrow q] \wedge q) \rightarrow p \qquad\qquad ([p \rightarrow q] \wedge p) \rightarrow q$$

Problem 11. Let p, q, and r denote logical statements. Use truth tables to determine if the following statements are tautologies.

$$([p \rightarrow q] \wedge [q \rightarrow r]) \rightarrow [p \rightarrow r] \qquad\qquad ([p \rightarrow \sim q] \wedge [r \rightarrow q] \wedge p) \rightarrow \sim r$$

> ***Logical Equivalence***
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> We say that logical statements p and q are *logically equivalent* provided the biconditional $p \leftrightarrow q$ is a tautology. We write $p \equiv q$ in this case.

Problem 12. Let p and q denote logical statements. If p is materially equivalent to q, is it also true that p is logically equivalent to q? Answer this question by considering the possible truth-values for p and q.