Now that we have developed some of the basic tools for deductive reasoning, we will turn attention to ways this method of reasoning can be used in mathematics[1].

Not everyone or even every culture understands "mathematics" in the same way. Most ancient civilizations developed very sophisticated bodies of mathematics, but their approach was empirical in nature. Mathematics was typically presented as collections of specific problems, and the reader was expected to generalize the problems to different situations through inductive reasoning.

For reasons that are not entirely clear, the ancient Greeks adopted a different approach between 500 BCE and 300 BCE. During this time, Greeks came to reject any mathematical idea that was not derived through valid deductive reasoning. This philosophical perspective on mathematics seems to have been unique in the ancient world; and over many centuries, it was adopted by European cultures as the preferred way to develop new "mathematics."

Following a road that is far too complicated and colorful to explore here, scientists and philosophers in Europe ultimately came to view "mathematics" as a family of logical deductions largely divorced from practical experience and empirical verification. This so-called "liberation" of mathematics was complete by the early twentieth century and led to one of the greatest flowerings of abstract reasoning the world has ever seen.

From the modern European perspective[2], a "mathematician" is a scientist-philosopher who adheres to the *axiomatic method*[3].

Axiomatic Method

The axiomatic method (*logico-deductive method*) sees mathematics as an essentially philosophical pursuit. Mathematical ideas are understood to be idealized constructs built up through valid arguments from a set of primitive terms left to their intuitive meaning and a set of fundamental statements about these terms that are assumed to be true. These fundamental statements are known as *axioms* or *postulates*.

We will devote the remainder of this course to investigating the axiomatic method as it applies to "sets."

Most mathematicians believe only three primitive terms are needed to define the term "set." In particular, it is common to assume the following terms or their synonyms are primitive:

- Object
- Collection
- Member of a collection

In the late nineteenth century, Georg Cantor, one of the great pioneers of abstract set theory, initially defined the term "set" as follows:

A *set* is any collection of objects having a rule for determining membership in the collection.

[1] The word "mathematics" comes from the Greek noun "*mathema*" which means "knowledge."
[2] The equity and social justice movement in the sciences is encouraging the adoption of a broader perspective.
[3] The word "axiom" comes from the Greek verb "*axioein*" which means "to deem worthy" or "to require."

Cantor's definition seems reasonable, but it has some strange consequences.

Problem 1. Consider the set T whose objects are all things which are teapots. Is T a member of the set T? Explain.

Problem 2. Consider the set U whose objects are all things which are *not* teapots. Is U a member of the set U? Explain.

Problem 3. Consider the following description of a "set" given by Bertrand Russell in 1901.[4]

Let S be the set of all sets which do not contain themselves as a member.

Does the set S contain itself as a member?

While a very careful modern definition of the term "set" has to some extent vanquished the issue presented by Problem 3, this approach is recondite at best; and it brings problems of its own. When working with sets, most mathematicians still use Cantor's definition, knowing full well that it leads to weirdness and simply taking care not to let the sets they are working with get "too big." This has come to be known as the *naïve* approach to sets. In the blissful spirit of naïve set theory, we end with a song.

A Set is a Set[5]

A set is a set.
(You bet! You bet!)
And nothing could not be a set,
You bet!
That is, my pet
Until you've met
My very special set!

If this were a set,
It'd be a threat,
And lead to conclusions that you'd regret,
And make you fret
And wet with sweat ---
This very special set!

Let A be the set of every U
That doesn't belong to U.
Then if A's in A, it's not in A
And if not, then what can you do?

So don't use the het-
Erological set
'Cause some things
Cannot be a set, my pet.
Or better yet,
Go out and get
The *class* of every set!

Ooooh, Bertrand

[4] This description is known as *Russell's Paradox*.
[5] Bruce Reznik, Mathematics Magazine, April 1993 Volume 66, No. 2. Sung to the tune of "Mr. Ed."