Suppose that ***G***=(G,*) and ***H*** = (H, #) are groups, and suppose *f* is a function *from* the set *G* *to* the set *H*. We say that *f* is an *isomorphism* provided the following conditions are satisfied.

- The function *f* is a bijection.
- The function *f* "preserves the operation;" that is, for all *a*, *b* in the set *G*, we have

$$f(a*b) = f(a)\#f(b)$$

When there is an isomorphism between the universes of two groups, we say the groups are *isomorphic*. Isomorphic groups have identical structure; that is, aside from the fact that the groups contain different elements, they are mathematically indistinguishable.

TASK 1: Suppose that ***G*** = (G,*) and ***H*** = (H, #) are isomorphic groups, and suppose that $f: G \longrightarrow H$ is an isomorphism from ***G*** to ***H***. Prove that the inverse function for *f* is an isomorphism from ***H*** to ***G***.

PROOF: Let $g: H \longrightarrow G$ be the inverse function for *f*. We know that $f\big(g(x)\big) = x$ for all $x \in H$, and we know $g\big(f(y)\big) = y$ for all $y \in G$.

Question: Why do we know that *g* is a bijection?

Let $a, b \in H$. We need to show that $g(a\#b) = g(a) * g(b)$. Since *f* is onto, we know there exist $x, y \in G$ such that $a = f(x)$ and $b = f(y)$.

Question: Why do we also know that $x = g(a)$ and $y = g(b)$?

There are several steps missing in the derivation below. What needs to be added to justify this derivation?

$$g(a\#b) = x * y = g(a) * g(b)$$

TASK 2: Suppose that ***G*** = (G,*) and ***H*** = (H, #) are isomorphic groups. If ***G*** is commutative, use the fact that an isomorphism exists between the groups to prove that ***H*** is also commutative.

TASK 3: Suppose that ***G*** = (G,*) and ***H*** = (H, #) are isomorphic groups, and suppose that $f: G \longrightarrow H$ is an isomorphism.

PART A: If *e* is the identity element for ***G***, prove that *f*(*e*) is the identity element for ***H***.

PART B: For all *a* in *G*, we have $f(a^{-1}) = [f(a)]^{-1}$. In other words, the inverse of *f*(*a*) in the group ***H*** is the image under *f* of the inverse for *a* in the group ***G***.

PART C: For all *a* in *G* and every positive integer *n*, prove we have $f(a^n) = [f(a)]^n$. (This proof requires induction.)

TASK 4: Explain why the dihedral group $\boldsymbol{D}_6$ is NOT isomorphic to the group $\boldsymbol{Z}_6$.

TASK 5: Explain why the Quaternion Group is NOT isomorphic to the dihedral group $\boldsymbol{D}_8$ of symmetries for the plus-sign.