Suppose $\boldsymbol{G} = (G,*)$ is any group, and suppose that $\boldsymbol{H}$ is a subgroup of $\boldsymbol{G}$. For any fixed element a in G, the set

$$a\boldsymbol{H} = \{a * x : x \in H\}$$

is called the *left coset* of $\boldsymbol{H}$ generated by a.

TASK 1: If a is an element of G, explain why we know $a \in aH$. (Therefore, every element of G appears in at least one left coset of $\boldsymbol{H}$.)

Suppose that $a, b \in G$, and suppose that $u \in a\boldsymbol{H} \cap b\boldsymbol{H}$. In the following argument, we will prove that $a\boldsymbol{H} = b\boldsymbol{H}$.

Proof: Since $u \in a\boldsymbol{H} \cap b\boldsymbol{H}$, we also know that $u \in a\boldsymbol{H}$ and $u \in b\boldsymbol{H}$. Therefore, there exist $x, y \in \boldsymbol{H}$ such that $a * x = u = b * y$. We will first prove $a\boldsymbol{H} \subseteq b\boldsymbol{H}$. To this end, suppose that $p \in a\boldsymbol{H}$.

TASK 2: Why do we know that $p = a * z$ for some $z \in \boldsymbol{H}$?

TASK 3: Why do we know that $a = b * (y * x^{-1})$?

TASK 4: Why is the following line of reasoning valid?

Consequently, there exist $w \in \boldsymbol{H}$ such that

$$p = a * z = (b * w) * z = b * (w * z) \in b\boldsymbol{H}$$

We may therefore conclude that $a\boldsymbol{H} \subseteq b\boldsymbol{H}$. , as desired.

TASK 5: Our next task will be to prove that $b\boldsymbol{H} \subseteq a\boldsymbol{H}$. Suppose that $q \in b\boldsymbol{H}$ and prove that $q \in a\boldsymbol{H}$.

The previous argument tells us that, if ***H*** is a subgroup of ***G***, and $a, b \in G$, then either $a\boldsymbol{H} = b\boldsymbol{H}$ or $a\boldsymbol{H} \cap b\boldsymbol{H} = \emptyset$. Combining this observation with Task 1, we can see that the set of all left cosets of ***H*** forms a partition of the group ***G***.

TASK 6: Let's return to the dihedral group $\boldsymbol{D}_8$.

Part (a): Construct the left cosets for the subgroup $\boldsymbol{H}_1 = \{S_i,\ S_{2R}\}$. There will be four distinct left cosets. Assign one of the colors white, pink, green, or blue to each coset.

Part (b): Construct the left cosets for the subgroup $\boldsymbol{H}_2 = \{S_i,\ {}_FS_{1R}\}$. There will be four distinct left cosets. Assign one of the colors white, pink, green, or blue to each coset.

TASK 7: Do the left cosets for the subgroup $\boldsymbol{H}_1$ form a group under the combining rule for subsets? In constructing the table, it helps to use colored index cards to represent each of the four left cosets.

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TASK 8: Do the left cosets for the subgroup $\boldsymbol{H}_2$ form a group under the combining rule for subsets? In constructing the table, it helps to use colored index cards to represent each of the four left cosets.

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