In the last activity, we discovered that only some subgroups of a group ***G*** can serve as the identity element for a quotient group from ***G***. In this activity, we will determine which ones will work.

TASK 1: Suppose ***G*** = (G,*) is any group, and suppose that ***H*** is a subgroup of ***G***. How would you define the *right coset* of ***H*** generated by g?

TASK 2: Construct the right cosets for the subgroup $\boldsymbol{H}_1 = \{S_i,\ S_{2R}\}$ of the dihedral group $\boldsymbol{D}_8$. How do the left and right cosets compare?

TASK 3: Construct the right cosets for the subgroup $\boldsymbol{H}_2 = \{S_i,\ {}_FS_{1R}\}$ of the dihedral group $\boldsymbol{D}_8$. How do the left and right cosets compare?

In the next tasks, we will fill in the gaps to a proof of the following result:

Let ***G*** =(G,*) be a group, and suppose that ***H*** is a subgroup of ***G***. If the subgroup ***H*** serves as the identity element for its family of left cosets under the combining rule for subsets, then a***H*** = ***H***a for all $a \in G$.

TASK 4: Explain why we know that $\boldsymbol{H} \circledast a\boldsymbol{H} = \{x * a * y : x, y \in \boldsymbol{H}\}$.

TASK 5: Use Task 4 to explain why $\boldsymbol{H}a \subseteq \boldsymbol{H} \circledast a\boldsymbol{H}$. (Try a special value for y.)

Now, if we assume that the subgroup ***H*** serves as the identity element for its family of left cosets under the combining rule for subsets, then $\boldsymbol{H} \circledast a\boldsymbol{H} = a\boldsymbol{H}$ for all for all $a \in G$, and we may conclude $\boldsymbol{H}a \subseteq a\boldsymbol{H}$.

Next, suppose that $a \in G$ and $x \in \boldsymbol{H}$. Since $\boldsymbol{H}a \subseteq a\boldsymbol{H}$, we know that $x * a \in \boldsymbol{H}a$; and we may conclude that there exist $y \in \boldsymbol{H}$ such that $x * a = a * y$.

TASK 6: Explain why we know that $a^{-1} * x * a \in \boldsymbol{H}$ for all $x \in \boldsymbol{H}$ and $a \in G$.

We are now ready to prove that $a\boldsymbol{H} \subseteq \boldsymbol{H}a$. To this end, suppose that $u \in a\boldsymbol{H}$. There exist $x \in \boldsymbol{H}$ such that $u = a * x$.

TASK 7: Use Task 3 to explain why we may conclude that $u * a^{-1} \in \boldsymbol{H}$.

Now, since $u * a^{-1} \in \boldsymbol{H}$, we know that $u = (u * a^{-1}) * a \in \boldsymbol{H}a$. Therefore, we may conclude that $\subseteq \boldsymbol{H}a$, as desired.

Let ***G*** = (G,*) be a group. A subgroup ***H*** of ***G*** is said to be *normal* provided *a**H*** = ***H**a* for all $a \in G$. We have proven the following result:

Let ***G*** = (G,*) be a group and let ***H*** be a subgroup of ***G***. If the left cosets of ***H*** form a quotient group from ***G***, then ***H*** must be normal.

Suppose now that ***H*** is a normal subgroup of a group ***G*** = (G,*). It turns out that the rule for combining left cosets of ***H*** can be made a lot simpler.

THEOREM: If ***H*** is a normal subgroup of a group ***G*** = (G,*), and $a, b \in G$, then it will always be true that

$$a\boldsymbol{H} \circledast b\boldsymbol{H} = (a * b)\boldsymbol{H}$$

We begin by showing that $a\boldsymbol{H} \circledast b\boldsymbol{H} \subseteq (a * b)\boldsymbol{H}$. Suppose that $u \in a\boldsymbol{H} \circledast b\boldsymbol{H}$.

TASK 8: Explain why we know there exist $x, y \in \boldsymbol{H}$ such that $u = a * (x * b) * y$.

TASK 9: Now, since ***H*** is normal, we know that $b\boldsymbol{H} = \boldsymbol{H}b$. Explain why there exist $z \in \boldsymbol{H}$ such that $x * b = b * z$.

TASK 10: Explain why $u = (a * b) * (z * y)$ and why this fact allows us to conclude $u \in (a * b)\boldsymbol{H}$.

Thus, $a\boldsymbol{H} \circledast b\boldsymbol{H} \subseteq (a * b)\boldsymbol{H}$ as desired.

Now, we will prove that $(a * b)\boldsymbol{H} \subseteq a\boldsymbol{H} \circledast b\boldsymbol{H}$. Suppose that $u \in (a * b)\boldsymbol{H}$.

TASK 11: Explain why that there exist $x \in \boldsymbol{H}$ such that $u = a * (b * x)$.

TASK 12: Now, since ***H*** is normal, we know that $b\boldsymbol{H} = \boldsymbol{H}b$. Explain why there exist $y \in \boldsymbol{H}$ such $u = (a * y) * (b * e)$, where *e* is the identity element for ***G***.

Consequently, we know $u \in a\boldsymbol{H} \circledast b\boldsymbol{H}$, and we may conclude $(a * b)\boldsymbol{H} \subseteq a\boldsymbol{H} \circledast b\boldsymbol{H}$, as desired.

THEOREM: Let ***G*** = (G,*) be a group, and suppose that ***H*** is a subgroup of ***G***. The set of left cosets of ***H*** forms a group under the rule for combining sets (with ***H*** as its identity element) if and only if ***H*** is a normal subgroup of ***G***.

PROOF: If the left cosets form a subgroup under the rule for combining sets, and ***H*** serves as the identity element, then we already know that ***H*** must be normal.

Conversely, suppose that ***H*** is normal. We already know that the combining rule for left cosets can be simplified, and we can use this to our advantage.

The combining rule for sets is an operation on the set of left cosets. To see why, suppose $a, b \in G$. We know that $a\boldsymbol{H} \circledast b\boldsymbol{H} = (a * b)\boldsymbol{H}$. Consequently, combining two left cosets gives us another left coset.

The set ***H*** serves as the identity element. To see why, note that $e\boldsymbol{H} = \boldsymbol{H}$. With this in mind, if $a\boldsymbol{H}$ is any left coset, we know

$$a\boldsymbol{H} \circledast \boldsymbol{H} = (a * e)\boldsymbol{H} = a\boldsymbol{H} \qquad \boldsymbol{H} \circledast a\boldsymbol{H} = (e * a)\boldsymbol{H} = a\boldsymbol{H}$$

If $a \in G$, then $a^{-1}H$ serves as the inverse for the coset $a\boldsymbol{H}$. To see why, simply observe that

$$a\boldsymbol{H} \circledast a^{-1}\boldsymbol{H} = (a * a^{-1})\boldsymbol{H} = e\boldsymbol{H} \qquad a^{-1}\boldsymbol{H} \circledast a\boldsymbol{H} = (a^{-1} * a)\boldsymbol{H} = e\boldsymbol{H}$$