Whenever we talk about a group, we must identify the underlying set of elements (called the *universe*) and the operation. We often use an ordered pair to do this. For example, the sentence

"Let $\boldsymbol{G} = (G,*)$ be a group."

means we are discussing a group whose universe is the set *G* and whose operation is *.

NOTE: If $\boldsymbol{G} = (G,*)$ is a group, and $a \in G$ then for any positive integer *n*,

$$a^n = a * a * a * \ldots * a$$

is always a member of *G*.

DEFINITION: Let $\boldsymbol{G} = (G,*)$ be a group with identity element $\boldsymbol{e}$. The *order* of an element $a \in G$ is the smallest positive integer *n* such that $a^n = \boldsymbol{e}$. If no such integer exists, we say that $a$ has *infinite order*.

TASK 1: Determine the order of each of the six symmetries of the equilateral triangle.

TASK 2: Determine the order of each of the eight symmetries of the plus-sign.

TASK 3: Determine the order of each of the eight members of the group $\boldsymbol{Z}_8$ .

TASK 4: What is the order of the positive integer 1 in the group $\boldsymbol{Z} = (Z, +)$ of integers under ordinary addition?

TASK 5: Let $\boldsymbol{G} = (G,*)$ be a group, and suppose that $\in G$ .

PART A: If *n* is a positive integer, how do you think we should define $a^{-n}$ ?

PART B: Based on your definition from Part A, how are the elements $a^{-n}$, $(a^{-1})^n$, and $(a^n)^{-1}$ related to each other? Prove your conjectures.

TASK 6: Let $\boldsymbol{G} = (G,*)$ be a group. Suppose that $a, b \in G$ have the following properties:

1) The element $a$ has order 2, and the element $b$ has order 4.
2) We have $a * b = b^3 * a$ .

PART A: Prove that we must have $a = b * a * b$ .

PART B: Prove that we must have $a * b^2 = b^2 * a$ .

TASK 7: Let $\boldsymbol{G} = (G,*)$ be a group and suppose that $,b \in G$ .

PART A: Prove that $(a * b)^2 = a^2 * b^2$ if and only if $a * b = b * a$ .

PART B: Prove that $(a * b)^{-1} = a^{-1} * b^{-1}$ if and only if $a * b = b * a$ .

TASK 8: If $\boldsymbol{G} = (G,*)$ is a non-commutative group, what is the relationship between the element $(a * b)^{-1}$ and the elements $a^{-1}$ and $b^{-1}$? Prove your conjecture.