DEFINITION: Let $\boldsymbol{G} = (G,*)$ be a group. A subset *H* of the universe *G* is called a *subgroup* of $\boldsymbol{G}$ provided *H* is a group in its own right when the combining rule * is restricted to the elements of *H*.

TASK 1: Let $\mathbf{5}Z$ represent the set of all integer multiples of 5. Let's prove that $\mathbf{5}Z$ is a subgroup of $\boldsymbol{Z} = (Z,+)$.

PART A: THE COMBINING RULE IS AN OPERATION ON $\mathbf{5}Z$.

Let a,b be members of $\mathbf{5}Z$*.*

Therefore, + is an operation on $\mathbf{5}Z$*.*

PART B: Do we need to worry about associativity of the operation on $\mathbf{5}Z$? Why or why not?

PART C: THE SET $\mathbf{5}Z$ HAS AN IDENTITY UNDER $+$.

*Let **e** be the identity element for the group* $\boldsymbol{Z} = (Z, +)$.

*Therefore, **e** serves as the identity element for* $\mathbf{5}Z$ *under* $+$.

PART D: EVERY MEMBER OF $\mathbf{5}Z$ HAS AN INVERSE UNDER $+$.

Let a be any member of $\mathbf{5}Z$*.*

Therefore, a has in inverse in $\mathbf{5}Z$ *under* $+$.

TASK 2: Let $\boldsymbol{G} = (G,*)$ be a group and let *H* be a subgroup of *G*.

PART A: If the identity of $\boldsymbol{G}$ is a member of *H*, prove that this element serves as the identity of *H*.

PART B: If an element of *H* serves as the identity for (*H*,*), prove that this element must also be the identity for $\boldsymbol{G}$.

TASK 3: Let $\boldsymbol{G} = (G,*)$ be a group and let H be a subgroup of G. Let a,b be members of H. Prove that b is the inverse of a in $(H,*)$ if and only if b is the inverse of a in $\boldsymbol{G}$.

According to our subgroup definition, we have to check all four group axioms to determine whether or not a given subset is a subgroup of some group. It would be nice to have a shorter list of conditions to check that will still guarantee that a subset is a subgroup. Fortunately, such lists exist.

TASK 4: Let $\boldsymbol{G} = (G,*)$ be a group and let H be a subset of G. Come up with a shorter list of conditions that are both necessary (that is, must be true if) and sufficient (that is, guarantee) H is a subgroup of $\boldsymbol{G}$.

TASK 5: Write a theorem based on your new list of conditions. If your theorem is an "if and only if" then state it this way. If it is not, then given a specific example to explain why.

THEOREM:

CLASS CONSENSUS:

TASK 6: Proof of the class consensus theorem:

TASK 7: Consider the following conjecture:

"Let $\boldsymbol{G} = (G,*)$ be a group and let H be a subset of G. If the combining rule * is an operation on H, then H is a subgroup of $\boldsymbol{G}$."

PART A: Find a specific counterexample for this conjecture.

PART B: There is a subtle flaw in this "proof" of the conjecture. What is the flaw?

Because the subset is closed, only elements from the subset appear in a row of the table for the subset. So every element of the subset must appear in each row. Since element A must appear in the row for element A, the identity must be in the subset. Then since the identity must appear in the row for element A, the inverse of A must be in the subset. Finally, since associativity holds for the whole group it must hold for the subset. Therefore the subset is a group and hence a subgroup.

PART C: There are situations where this argument would be valid. What are those situations?

PART D: How could we modify the conjecture itself so that the argument above would serve as a proof?