TENTH GRADED HOMEWORK ASSIGNMENT

Let X be a nonempty set. Recall that the system $\wp_X = (P_X\,,\,\circ)$ of permutations on X under the combining rule of function composition is always a group. The identity element will be the identity function, and for each bijection $\in P_X$, the function inverse for f serves as the group inverse for f .

There are some notational conventions we use when working with the group of permutations on a *finite* set X. First, if X contains exactly n elements, then we assume

$$X = \{1, 2, 3, \ldots, n\}$$

and we write P_n to represent the set of permutations on X. Note that P_n will contain exactly

$$n! = 1 \cdot 2 \cdot \ldots \cdot n$$

members. If $f \in P_X$, then the range of f is just a rearrangement of the members of X. We use a convenient *tabular notation* to represent the inputs and outputs of f.

As an example, suppose that $f \in P_4$ is defined by the following rule:

$$f(1) = 3 \qquad f(2) = 1 \qquad f(3) = 2 \qquad f(4) = 4$$

Writing this function in tabular notation would give us

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

This matrix-like notation is just a simple way to express the assignment process without having to write all of the accompanying function notation. In tabular notation, the top row always represents the inputs (written in ascending numerical order), while the bottom always represents the corresponding outputs.

We can compute the composition of two permutations from P_n using tabular notation as well. We only need to remember that function composition is read *from right to left*. Suppose $f, g \in P_4$ are defined by

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \qquad g : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

The composition $g \circ f$ would be

Move right to left

$$g \circ f : \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

In words, we know that $f(1) = 3$, and we know that $g(3) = 2$. Therefore, we know $g\big(f(1)\big) = 2$. Each column of the tabular notation for $g \circ f$ is filled in this way.

Problem 1: Use tabular notation to compute the composition $f \circ g$ and $g \circ f$ for the following permutations in P_6.

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 6 & 5 \end{pmatrix} \qquad g : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$$

Problem 2: Determine the inverse in the group $\wp_8 = (P_8\,,\ \circ)$ for the permutation h shown below and write the inverse in tabular notation.

$$h : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 2 & 6 & 1 & 8 & 3 & 4 \end{pmatrix}$$

Problem 3: Show that the permutation f from Problem 1 has order 6 in the group $\wp_6 = (P_6\,,\ \circ)$. Does this mean that this group is cyclic? Explain your answer.

Let $X = \{1, 2, 3, \dots, n\}$ and let $\{a_1\,,\ a_2, \dots\,,\ a_k\} \subseteq X$. A permutation $f \in P_n$ is called a *k-cycle* provided the following conditions are met:

- If $b \notin \{a_1\,,\ a_2, \dots\,,\ a_k\}$ then $(b) = b$.
- We have $f(a_1) = a_2\,, f(a_2) = a_3\,, f(a_3) = a_4\,, \dots, f(a_k) = a_1$.

None of the permutations used in Problems 1 – 3 above are k-cycles, but each of the following is a k-cycle.

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 5 & 2 & 6 \end{pmatrix} \qquad g : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

3-cycle in P_6 2-cycle in P_4

It is customary to compress the tabular notation even further for permutations that are *k*-cycles. For example, the *k*-cycles shown above can be written

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 5 & 2 & 6 \end{pmatrix} = (245) \qquad g : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = (13)$$

We understand the expression to (245) mean $f(2) = 4, f(4) = 5, f(5) = 2$, while $f(1) = 1$, $f(3) = 3,$ and $f(6) = 6$.

Problem 4: Which, if any, of the following 3-cycles is the same as $f : (245)$ above? How did you decide?

$$g : (524) \qquad h : (425) \qquad j : (452)$$

Problem 5: In the problem below, expand the k-cycles into tabular notation and compute the composition. Assume all cycles are members of P_6.

$$f : (216) \circ (36) \circ (214)$$

Problem 6: Show that the 3-cycle $g : (612)$ is the group inverse of the 3-cycle $f : (216)$ in the group $\wp_6$.

Two permutations are *disjoint* provided the sets of elements that they move are disjoint. Every permutation that is not a *k*-cycle can be written as a composition of disjoint *k*-cycles. For example,

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 2 & 6 & 1 & 8 & 3 & 4 \end{pmatrix} = (15) \circ (273) \circ (468)$$

Problem 7: Take the permutation f you computed in Problem 5 and write it as a composition of disjoint cycles.