THIRTEENTH GRADED HOMEWORK ASSIGNMENT

Problem 1. Suppose that ***G*** = (G,*) and ***H*** = (H, #) are isomorphic groups, and suppose that $f: G \longrightarrow H$ is an isomorphism. Let *n* be a positive integer. Use Task 3 (B) and (C) of Activity 3 to prove $f(a^{-n}) = [f(a)]^{-n}$. (Therefore, *f* preserves *all* powers of *a*.)

Proof: We know that $f(a^{-1}) = [f(a)]^{-1}$, we know that $f(a^n) = [f(a)]^n$ for any positive integer *n*, and we know that for any positive integer *n*, $a^{-n} = (a^{-1})^n$. Combining these facts gives us

$$f(a^{-n}) = f([a^{-1}]^n) = [f(a^{-1})]^n = ([f(a)]^{-1})^n = [f(a)]^{-n}$$

Problem 2. Suppose that ***G*** = (G,*) and ***H*** = (H, #) are isomorphic groups, and suppose that $f: G \longrightarrow H$ is an isomorphism from ***G*** to ***H***. Suppose that $a \in G$, has order *n*.

Part (a) Use Task 3 (A) and (C) of Activity 3 to help show that the order of *f*(*a*) is at most *n*.

Proof: Let *i* be the identity element for ***H*** and let *e* be the identity element for ***G***. We know that $i = f(e) = f(a^n) = [f(a)]^n$. Since the order of $f(a)$ is the smallest positive integer *m* such that $[f(a)]^m = i$, we may therefore conclude that the order of $f(a)$ must be no larger than *n*.

Part (b) Let *m* be the order of *f*(*a*) and let $g: H \longrightarrow G$ be the inverse function for *f*. Use the fact that *g* is also an isomorphism, along with Task 3 (A) and (C), to show that the order of *a* is at most *m*. (Therefore, we must have *n* = *m*.)

Proof: Let *i* be the identity element for ***H*** and let *e* be the identity element for ***G***. We know that

$$e = g(i) = g([f(a)]^m) = g(f(a^m) = a^m$$

Since the order of a is the smallest positive integer *n* such that $a^m = e$, we may therefore conclude that the order of a must be no larger than *m*.

Problem 2 tells us that an isomorphism between groups must preserve the order of elements that have finite order. (It must preserve infinite order as well, but a different proof is needed for this.)

Problem 3. Suppose that $\boldsymbol{G}$ = (G,*) is an infinite cyclic group, and suppose that *a* is a generator for $\boldsymbol{G}$. We know

$$G = \{a^n : n \in \mathbb{Z}\}$$

Use this fact to construct an isomorphism from the group $\boldsymbol{Z} = (\mathbb{Z}, +)$ of integers under addition to the group $\boldsymbol{G}$. You will need the results from Homework Assignment 8 to prove that your function is an isomorphism.

Proof: The natural choice is the function $f : \mathbb{Z} \to G$ defined by $f(m) = a^m$. It is easy to see that f preserves the operation. Indeed, according to the definition of powers in a group, we know

$$f(m+n) = a^{m+n} = a^m * a^n = f(m) * f(n)$$

It is also clear that f is onto. Indeed, if $a^m \in G$, then by definition, $f(m) = a^m$. It is not so apparent that f is one-to-one; however, we established this fact in Homework Assignment 8. Indeed, if $f(m) = f(n)$, then we know $a^m = a^n$. Since $\boldsymbol{G}$ is an infinite cyclic group with *a* as a generator, we know that *a* has infinite order. Therefore, we know $m = n$ from Homework Assignment 8.

Problem 4: Explain why the dihedral group $\boldsymbol{D}_6$ is NOT isomorphic to the group $\boldsymbol{Z}_6$.

The group $\boldsymbol{D}_6$ is not commutative, while the group $\boldsymbol{Z}_6$ is commutative. Also, the group $\boldsymbol{Z}_6$ contains two elements of order 6, while the group $\boldsymbol{D}_6$ contains no element of order 6.

Problem 5: Explain why the Quaternion Group is NOT isomorphic to the dihedral group $\boldsymbol{D}_8$ of symmetries for the plus-sign.

The dihedral group contains three elements of order 4, namely the three non-identity rotations. The Quaternion group contains five elements of order 4. Since an isomorphism must preserve the order of elements, it is not possible construct an isomorphism between these groups.