EIGHTH GRADED HOMEWORK ASSIGNMENT

Problem 1. Let $\boldsymbol{G} = (G,*)$ be a group with identity element ***e***, and suppose that $a \in G$. Suppose that *j* and *k* are positive integers such that *j* < *k*.

PART A: If $a^j = a^k$, what can we say about the element a^{k-j} ? Justify your answer.

If , $a^j = a^k$ then we know from the laws of exponents that $a^{k-j} = e$.

PART B: If $a^j = a^k$, use your answer to Part A to explain why the order of the element *a* must be strictly less than *k*.

By definition, the order of a is the smallest positive integer n such that $a^n = e$. Since we know that $a^{k-j} = e$, we must conclude that the order a of is *at most k-j*. Now, since 0< *j* < *k*, we know that *k-j* < *k*, and we therefore know that the order of a is strictly less than *k*.

Problem 2. Let $\boldsymbol{G} = (G,*)$ be a group with identity element ***e***, and suppose that $a \in G$ has order *n*. Use Problem 1 (B) to explain why it is not possible to have $a^j = a^k$ whenever 0 ≤ *j* < *k* < *n*.

We know that *k – j* $\leq k < n$. We also know that n is the smallest positive integer such that $a^n = e$. Now, if we have $a^j = a^k$, we know that $a^{k-j} = e$. This forces us to conclude that the order of a is strictly smaller than *n* --- contrary to assumption.

Problem 3. Let $\boldsymbol{G} = (G,*)$ be a group with identity element ***e***, and suppose that $a \in G$. In this exercise, we address the following question.

Suppose *x* and *y* are positive integers. When does $a^x = a^y$ guarantee that *x* = *y*?

PART A: If *a* has infinite order, use Problem 1 (B) to prove that $a^x = a^y$ always guarantees that *x* = *y*.

If we ever have $a^x = a^y$ for *x* < *y*, then we know that the order of a must be strictly less than *y* --- contrary to assumption.

PART B: Suppose *a* has finite order *n* and suppose that *x* and *y* are positive integers such that that $kn \leq x, y < (k+1)n$ for some integer *k*. (That is, *x* and *y* are both between consecutive integer multiples of *n*.)

Part (i) Use the Division Algorithm to show there exist integers *r, s* such that

- $0 \leq r, s < n$
- $x = kn + r$ and $y = kn + s$

We know that there exist unique integers *a ,b, r,* and *s* such that $0 \leq r, s < n$, and

$$x = an + r \text{ and } y = bn + s$$

Now, we also know that $kn \leq x < (k+1)n$. If $a < k$, then we know that $a \leq k - 1$; and the fact that $r < n$ therefore tells us

$$x = an + r < (k-1)n + n = kn$$

This is contrary to assumption, so we must have $k \leq a$. On the other hand, if $k < a$, then we know that that $a \geq k + 1$; and the fact that $r \geq 0$ therefore tells us

$$x = an + r \geq (k+1)n + 0 = (k+1)n$$

This is also contrary to assumption, so we must have $k = a$. By a similar argument, we can establish that $k = b$ as well.

Part (ii) Use Problem 3 (A) from Homework Assignment 6 along with Problem 2 from this assignment to show that $a^x = a^y$ implies *x* = *y*.

Suppose that $kn \leq x, y < (k+1)n$. Part (i) tells us that

$$a^x = a^{kn} * a^r = a^r \text{ and } a^y = a^{kn} * a^s = a^s$$

where $0 \leq r, s < n$. However, we therefore know from Problem 2 that it is not possible to have $a^r = a^s$ unless we have *r* = *s*. Therefore, $a^x = a^y$ implies that

$$x = kn + r = kn + s = y$$