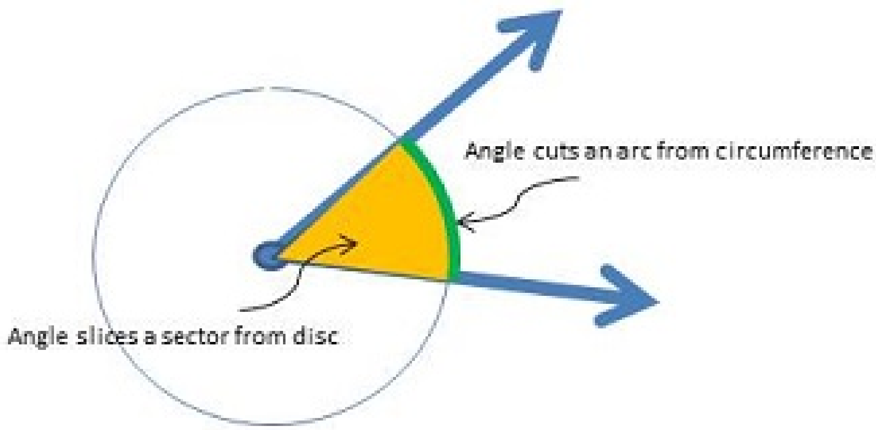
# TRIGONOMETRY

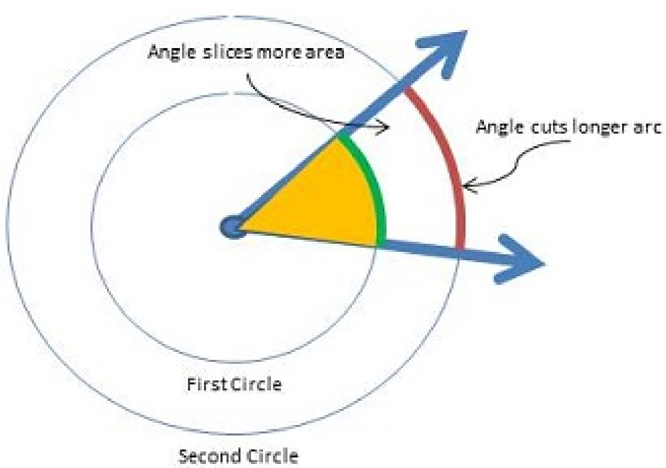
## 1 Investigation 1 — Measuring Angles

An *angle* is the "gap" formed by two rays that have the same endpoint. This endpoint is called the *vertex* of the angle. Every pair of rays that have the same endpoint actually make *two* angles, as the diagram below shows.


We have two standard ways to measure angles, and to understand something about these ways, it is helpful to let the vertex of your angle be the center of a circle. It does not matter what the radius of this circle is.

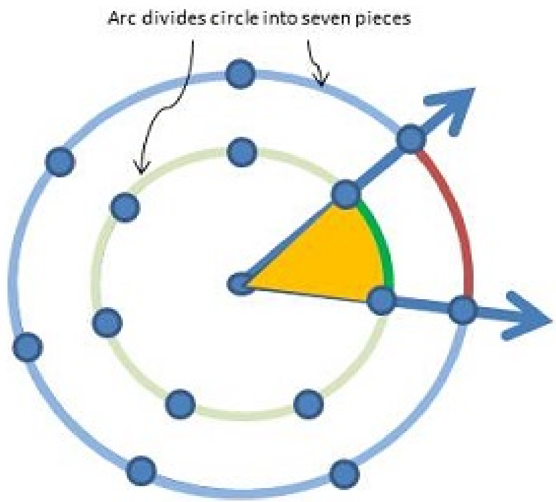



The angle shown above slices a sector from the interior of the circle, and it also cuts an arc from the circumference of the circle. We could measure either of these attributes for the angle, so we might consider using either one as a way to measure the angle. However, there is a big problem with this approach — what if we use a *different* circle instead?



As the figure above shows, if we use a bigger circle, the *same* angle slices a larger sector and cuts a longer arc. For this reason, we don't use attributes that come directly from a circle to measure angles. We adopt a more subtle approach.

First, notice that if we mark off the arcs cut by the angle around their circles, we can see that they each divide their circles into the same number of pieces.




Although the arcs cut by this angle from each circle have different lengths, they are both one-seventh the circumference of the circles they are cut from. This observation points out something critical about the relationship between the circles we draw and the arcs cut by the angle from each one:

**Fundamental Property of Angles and Arcs**

- If the vertex of an angle is at the center of two circles, the arcs cut by the angle will be the same fraction of their circle's circumference.

Suppose we let $R$ be the radius of the outer circle in the diagram above (measured in inches, let's say). The circumference of the circle is a constant multiple of the radius; in fact, the circumference is $C = 2\pi R$ inches. This means that the length $S$ of the arc cut by this angle from the circle is also some multiple of $R$. What mulitple is it? We know that $S$ is one-seventh the circumference, so

$$S = \frac{1}{7} \cdot 2\pi R \text{ inches} \qquad \Longrightarrow \qquad S = \left(\frac{2\pi}{7}\right) \cdot R \text{ inches}$$

Since both $S$ and $R$ are measured in inches, the fraction $\frac{2\pi}{7}$ has no units associated with it. We call this unitless number the *radian measure* of the angle shown in the diagram above.

**RADIAN MEASURE**

- Suppose the vertex of an angle is at the center of a circle of radius $R$, measured in whatever units of length you wish. The angle will cut an arc of length $S$ from the circumference of this circle (measured in the same units as $R$). The *radian measure* of this angle is the fraction (or multiple) of $R$ represented by the arc length $S$. If we let $\theta$ be this fraction or multiple, we have the following three relationships that define the radian measure of this angle.

$$S = \theta \cdot R \qquad \theta = \frac{S}{R} \qquad R = \frac{S}{\theta}$$

**Problem 1.** Suppose the vertex of an angle is the center of a circle having a two foot radius. If the arc cut from this circle by the angle is five inches long, what is the radian measure of this angle?

**Problem 2.** Suppose the radian measure of an angle is $\theta = 2.31$. If the vertex of this angle is the center of a circle with a three-meter radius, how long is the arc cut by this angle?

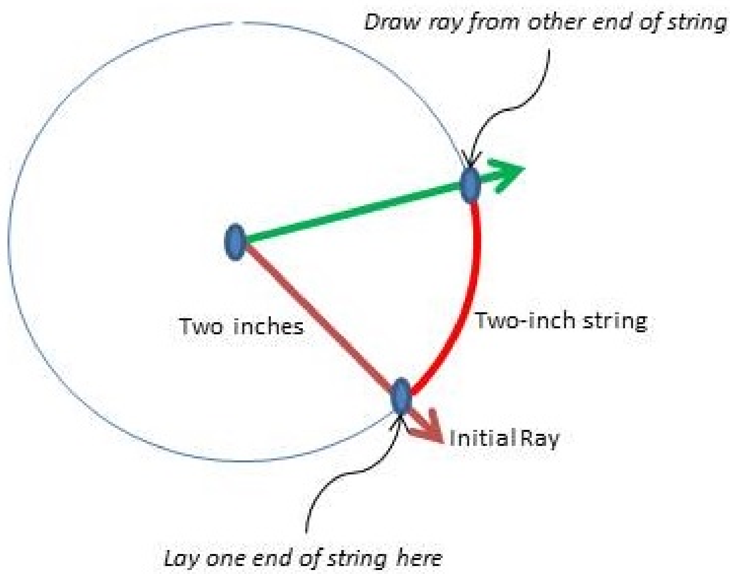
**Problem 3.** Suppose the radian measure of an angle is $\theta = \dfrac{7\pi}{8}$. If the vertex of this angle is the center of a circle, and the angle cuts an arc three feet long from this circle, what is the length of the radius?

**Problem 4.** The vertex of an angle is at the center of a circle whose radius is measured in miles, and the arc it cuts from this circle is exactly one-fourth the circle's circumference. What is the radian measure of this angle?

The radian measure of an angle is unitless. However, it is sometimes helpful to have some type of unit to work with when we are dealing with radian measure, so we invent a special unit just for this purpose. One *radian* is defined to be the angle required to cut an arc from any circle that is exactly equal in length to the circle's radius. This special angle is called a `rad` — the arc cut from any circle by a `rad` is equal to one radius length. The radian measure of any angle can be found by counting the number of `rad`'s enclosed between the two rays that define the angle.

**Example 1** *What does a one- radian angle look like?*

**Solution.** To see what such an angle looks like, let's work with a circle whose radius is two inches. When its vertex is the center of this circle, one radian will cut an arc from the circle that is exactly two inches long. We can draw one radius for this circle, then cut a two-inch piece of string and lay it on the perimeter of the circle so that one end lies at point where the radius we drew intersects the circle. We then draw another ray from the other end of the string back to the center. The resulting "gap"is a one-radian angle.

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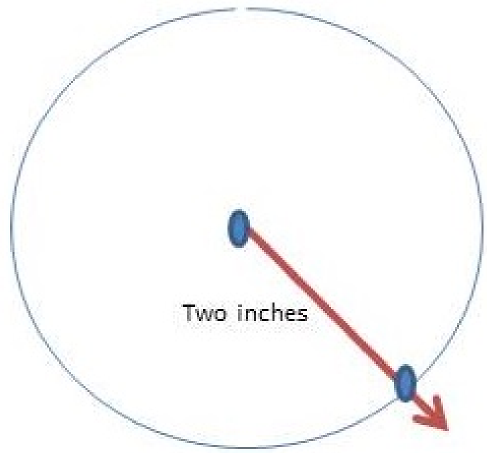
**Problem 5.** What does it mean to say that an angle has a measure of 2.67 `rad`?

**Problem 6.** Assume the circle below has a radius of two inches. Using the initial ray given and a piece of string, draw an angle whose measure is exactly 3 `rad`.




## DEGREE MEASURE

*Degree measure* is another, much older way to measure angles. It is not used much in mathematics, physics, or engineering for a number of reasons, most of which have to do with calculus. It is still widely used in navigation, surveying, and meteorology however; and it is therefore worth mentioning. In degree measure, the circumference of a circle is is divided into 360 equal segments, and the number of these segments enclosed between the rays that define the angle is called the *degree measure* of that angle. To measure an angle in degrees, we imagine its vertex is at the center of a circle (of any radius), and we simply count the number of degrees enclosed between the rays that define the angle. It is traditional to use a superscipted circle on the degree measurement to indicate that the angle measurement is in degrees. For example

Angle measure of 38.92 degrees is written $38.92^\circ$

**Example 2** *What is the radian measure of an angle whose degree measure is* $120^\circ$*?*

**Solution.** Suppose the vertex of our angle is the center of a circle of radius $R$ (measured in inches, let's say). To answer this question, we first observe that 120 degrees is

$$\frac{120}{360} = \frac{1}{3}$$

the total degree measure of the circle. Consequently, our angle will cut an arc that is one-third the total circumference of the circle. The length of this arc will therefore be $S = \frac{1}{3} \cdot (2\pi R)$ inches. The radian measure of this angle is

$$\theta = \frac{2\pi}{3} \text{ rad}$$

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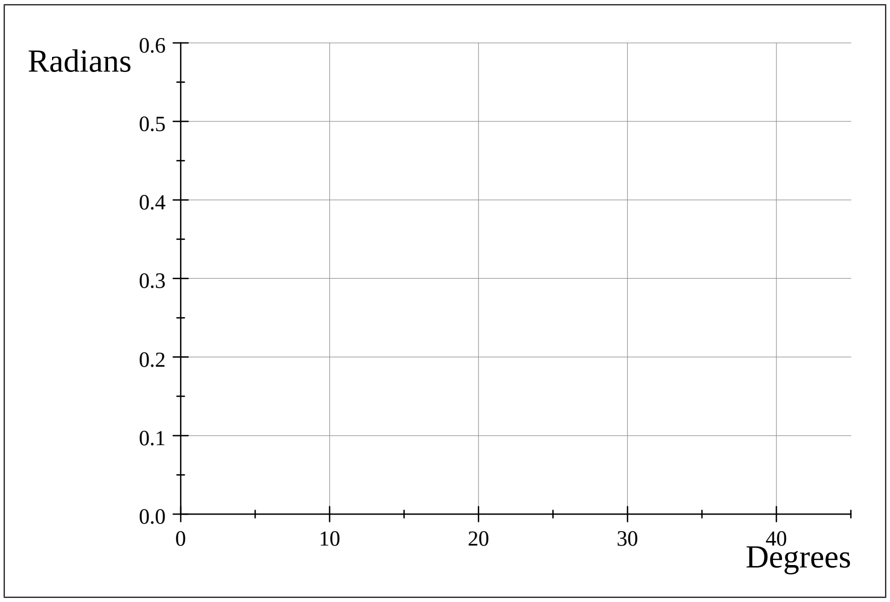
**Problem 7.** Use the method of the previous example to fill in the table below. Assume the vertex of each angle is the center of a circle of radius $R$ inches.

| Degree Measure of Angle | Fraction of 360° | Length of Arc | Radian Measure |
|---|---|---|---|
| 0° | | | |
| 15° | | | |
| 30° | | | |

**Problem 8.** On the grid provided, plot the radian measure of the angles above as a function of their degree measure. What do you notice about these points?




**Problem 9.** Let $d$ be the degree measure of an angle, and let $\theta$ be its radian measure. Use the table above to construct a function $f$ that gives $\theta$ in terms of $d$.

**Problem 10.** Let $g$ be the function that reverses $f$. Find a formula for the function $g$.

**Problem 11.** What is the meaning of the equation $\theta = f(d)$? What is the meaning of the equation $d = g(\theta)$?

**Problem 12.** Use your work in Problems 9, 10, and 11 to explain why the following conversion formulas work.

- If $d$ is the degree measure of an angle, and $\theta$ is the radian measure of the same angle, then

$$\theta = \frac{\pi}{180} \cdot d \qquad\qquad d = \frac{180}{\pi} \cdot \theta$$

**Supplemental Homework Problems**

1. Explain what it means in terms of arc length and radius for an angle to have radian measure

   (a) $1$ `rad` (b) $2.1$ `rad` (c) $\pi$ `rad` (d) $\frac{\pi}{6}$ `rad`

2. Convert the following angle measures from degree measure to radian measure.

   (a) $37°$ (b) $310°$ (c) $25.75°$ (d) $1°$

3. Convert the following angle measures from radian measure to degree measure.

   (a) $1$ `rad` (b) $3$ `rad` (c) $\pi$ `rad` (d) $\frac{\pi}{4}$ `rad`

4. Three angles each have their vertex at the center of a circle having a radius of twenty-one inches. The measures of these angles are given below. Determine the length in inches of the arc cut from the circle by each angle.

   (a) $1$ `rad` (b) $3$ `rad` (c) $\frac{3\pi}{2}$ `rad` (d) $120°$

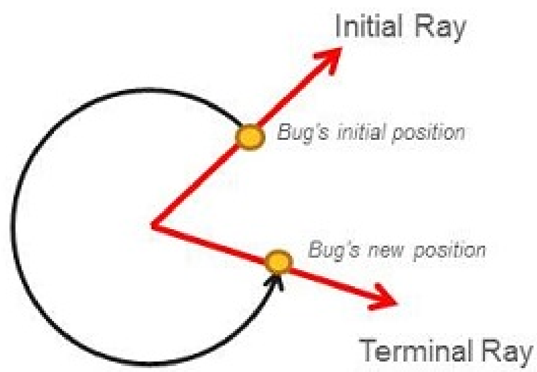
5. Mildred walks four hundred feet around a circular track having a one-hundred foot radius. What is the radian measure of the angle Mildred creates as she walks from her starting point to her finishing point? What is the degree measure of this angle?

6. Jose boards a ferris wheel having a radius of fifty feet. The ferris wheel starts turning and Jose rotates through a $217^\circ$ angle before the wheel stops to allow more passengers to get on. How far in feet did Jose travel around the wheel?

## 2 Investigation 2 — A Second Look at Angle Measure

In the sciences, angles are often used to represent rotations. Angles used in this way are sometimes called *rotation* angles. In this investigation, we will explore an example of how rotation angles are interpreted and used.

A bug lands on the tip of one blade on a ceiling fan having a two-foot radius. Jade turns on the fan, and it begins to rotate slowly counterclockwise at a constant speed. As the fan blades rotate, the bug will move from its initial loctaion counterclockwise along the circumference of a two-foot radius circle. Imagine a ray extending from the center of the fan through the bug's initial position. If we took a photo of the fan (freezing the motion) and drew a ray from the center of the fan to the bug's new position, these two rays would form an angle.




The ray passing through the bug's initial position is called the *initial ray* of the angle formed. The ray passing through the bug's new position is called the *terminal ray* of the angle formed. The bug's path is shown as the directed arc in the diagram above. (This is called a *direction arc* for the angle). The arrow on a direction arc always points from the initial ray to the terminal ray in the direction of the rotation.

**Problem 1.** Suppose the bug travels nine feet around the circumference of the circle from its initial position to its new position.. What is the radian measure of the angle in the diagram above?

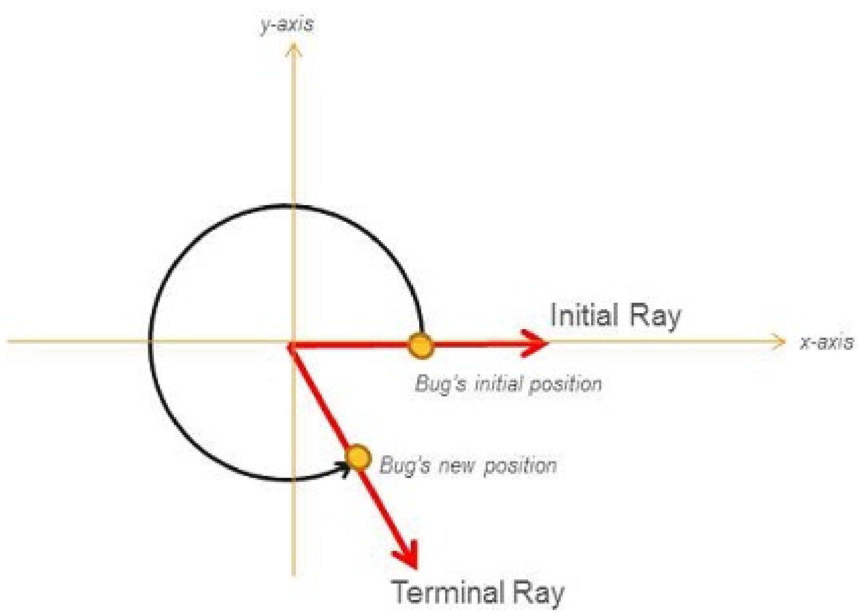
**Problem 2.** Suppose a wad of gum is stuck to the same fan blade one foot from the center of the fan. Through what distance will the wad of gum travel as the bug goes from its initial position to its new position?

**Problem 3.** The *angular speed* of the bug is defined to be the number of radians (radius lengths) the bug travels in one unit of time. (The units of angular speed are `rad`'s per unit of time.) If it took the bug three seconds to travel from its initial position to its new position, what is its angular speed?

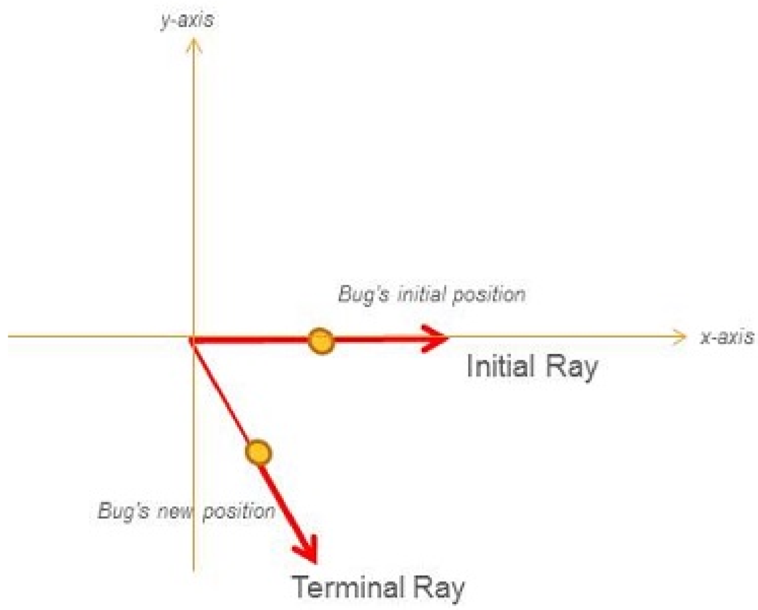
Sometimes we need to locate a point on one ray of an angle with greater precision than we used in the diagram above. When this is required, we imagine a rectangular grid is positioned so that the following conditions are met.

- The vertex of the angle lies at the origin of the grid.
- The initial ray of the angle lies on the positive half of the horzintal axis.

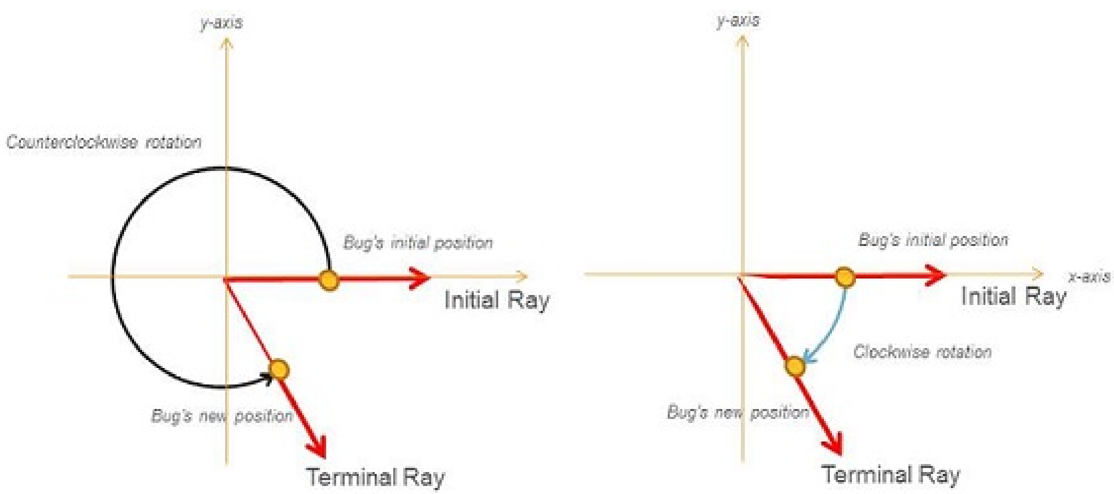
When these conditions are met, we say that the angle is in *standard position* on the grid. When an angle is in standard position on a grid, we have a systemaitc way of identifying points on both rays defining the angle. The diagram below shows the bug's rotation angle placed in standard position.



Here is the previous diagram showing the angle in standard position made by the bug as it rotates from its initial position to its new position, only this time with no direction arc to indicate which direction the bug traveled.




There are two ways the bug could have gotten from its initial position to its new position by traveling on the tip of the fan blade — the blade could have rotated in the *counterclockwise* direction, or it could have rotated in the *clockwise direction*. These two possibilities are illustrated in the figure below.




Compare the arcs cut by this angle from the two-foot circle when we rotate in the counterclockwise direction versus the clockwise direction. It is clear from the diagrams that these arcs have different lengths, and this means these arcs give rise to different radian measures for each of the two rotation angles.

In order to tell the difference between the clockwise-oriented radian measure and the counterclockwise-oriented radian measure for a rotation angle, we adopt the following convention:

**Distinguishing Rotation Direction**

*Radian measure will be positive for counterclockwise rotation and negative for clockwise rotation.*

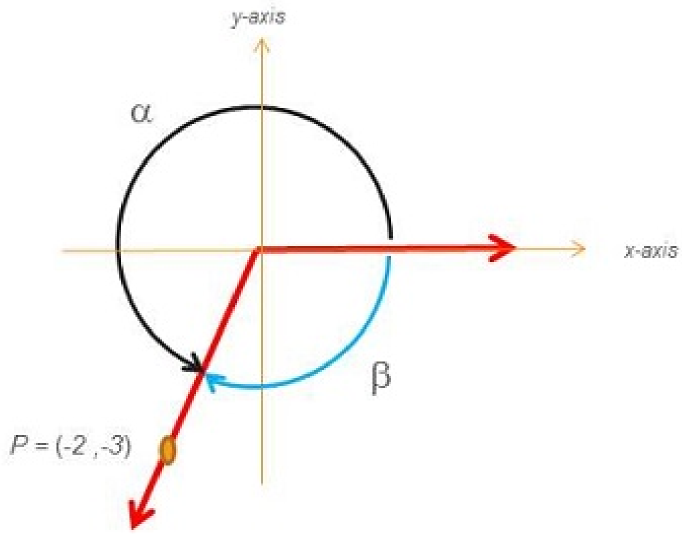
In other words, when you are told that an angle has radian measure $-3.17$ `rad`, this means that the measure represents a rotation through 3.17 radius lengths *in the clockwise direction* along the circumference of any circle centered on the angle vertex.

**Problem 4.** As the fan blade rotates, it traces out a circle with a two-foot radius. Suppose the bug rotates two-thirds of the way around this circle in the clockwise direction. What is the clockwise-oriented radian measure for this rotation angle?

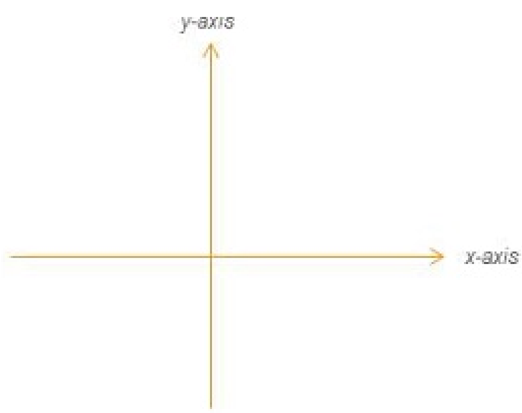
**Problem 5.** If the bug rotates two thirds of the way around the circle in the clockwise direction, how far around the circle in the *counterclockwise* direction would the bug have to rotate to form the same angle? What is the *counterclockwise*-oriented radian measure for this rotation angle?

**Problem 6.** Suppose the bug rotates through an angle having radian measure $-\frac{5\pi}{8}$ `rad`. What is the *counterclockwise*-oriented radian measure for this angle?

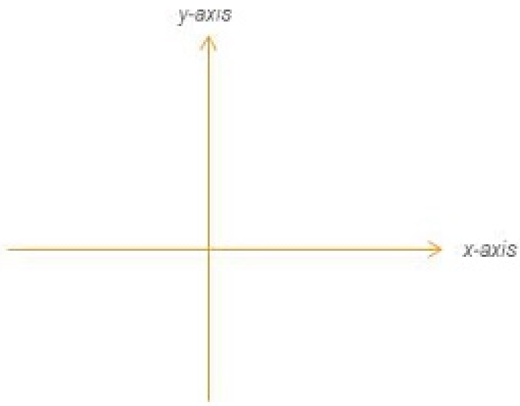
**Problem 7.** Consider the two rotation angles shown in the diagram below, one with counterclockwise-oriented radian measure $\alpha$ and the other with clockwise-oriented radian measure $\beta$. Explain why we have $\alpha - \beta = 2\pi$ `rad`. (Remember, $\beta$ is negative.)

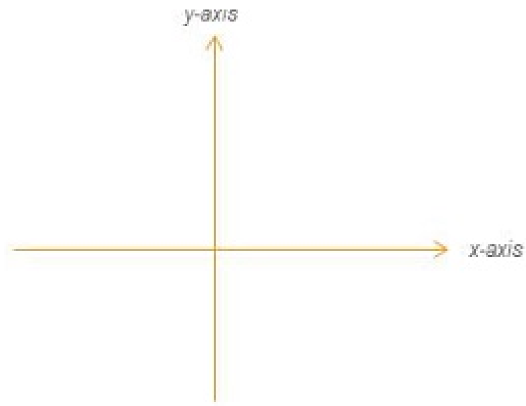


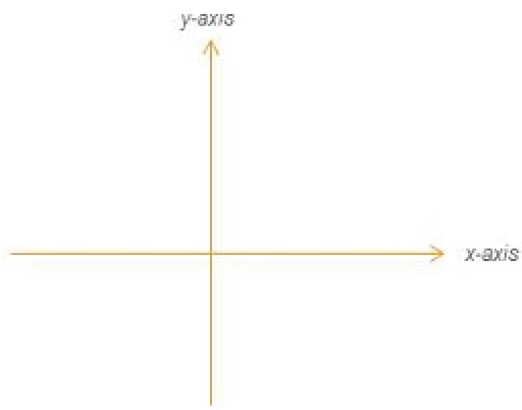

**Problem 8.** Let's return to the fan problem again. We know that an angle measure of $2\pi$ `rad` represents represents one complete rotation around the two-foot circle in the counterclockwise direction (so the bug's initial position and new position are the same). Suppose you are told that the bug has rotated through an angle having radian measure $3\pi$ `rad`. How would you explain the bug's motion around the fan? Draw this angle and its direction arc on the grid below.



**Problem 9.** On the first grid below, draw a rotation angle in standard position that has radian measure $\frac{\pi}{4}$ `rad`. On the second grid, draw a rotation angle in standard position having radian measure $-\frac{7\pi}{4}$ `rad`, and on the third grid, draw a rotation angle in standard position that has radian measure $\frac{9\pi}{4}$ `rad`. Do you notice anything about these three angles?




**Problem 10.** Suppose that someone points out to you that

$$\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi \qquad -\frac{7\pi}{4} = \frac{\pi}{4} - 2\pi$$

How could you use this to explain what you noticed in Problem 9? Think in terms of rotations.

Two rotation angles are said to be *coterminal* if they share the same initial ray and terminal ray. Coterminal angles represent different ways to get *from* one fixed point on a circle *to* another fixed point on the same circle. The two rotation angles shown in Problem 7 above are coterminal. The three rotation angles you drew in Problem 9 are also coterminal.

*There are infinitely many rotation angles that are coterminal with any given rotation angle.*

Coterminal rotation angles all differ by a series of complete rotations. Consequently, all coterminal angles will have radian measures that differ by integer multiples of $2\pi$.

Of all the rotation angles coterminal with a given rotation angle, there is always one that has the *smallest positive radian measure*. This particular angle is called the *fundamental rotation angle* associated with the given angle. In Problem 7 above, the rotation angle with measure $\alpha$ is the fundamental rotation angle. The fundamental rotation angle associated with a given rotation angle represents the smallest rotation *in the counterclockwise direction* around a circle that gets you *from* a fixed point on the initial ray of the given angle *to* a fixed point on the terminal ray of the given angle.

**Example 3** *What is the measure of the fundamental rotation angle associated with the angle having radian measure* $\alpha = -\dfrac{23\pi}{8}$ $\texttt{rad}$*?*

**Solution.** The measure of the given angle is clockwise-oriented. To find the smallest positive measure for this angle, we simply *add* $2\pi$ $\texttt{rad}$ to this measure repeatedly until we first get a positive number. Each value we get along the way will be another radian measure for this angle.

$$-\frac{23\pi}{8} + 2\pi = \frac{-23\pi + 16\pi}{8} = -\frac{7\pi}{8} \quad \text{The measure of another coterminal angle}$$

$$-\frac{7\pi}{8} + 2\pi = \frac{-7\pi + 16\pi}{8} = \frac{9\pi}{8} \quad \text{The measure of another coterminal angle}$$

We just went positive with the last addition of $2\pi$ $\texttt{rad}$, so $\theta = \dfrac{9\pi}{8}$ $\texttt{rad}$ is the measure of the fundamental rotation angle. Note that the measure $\alpha$ of the given angle differs from the fundamental measure by two complete *clockwise-oriented* rotations.

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**Problem 10.** Suppose the measure for an angle is $\theta = 3.50$ $\texttt{rad}$. The value $\alpha = 22.35$ $\texttt{rad}$ is the radian measure of another angle coterminal with the one have measure $\theta$.. By how many complete rotations (and in what direction) does the measure $\alpha$ differ from the measure $\theta$?

**Problem 11.** What is the measure of the fundamental rotation angle associated with the angle having measure $\alpha = \dfrac{43\pi}{6}$ $\texttt{rad}$? By how many complete rotations (and in what direction) does $\alpha$ differ from the fundamental measure?

**Supplemental Problems**

1. Suppose the angle defined by the initial position and new position of the bug has degree measure 38°. What is the length of the arc the bug traveled through?

2. Suppose the bug travels through an arc of five feet as it moves from its initial position to its new position counterclockwise in the two-foot radius fan. What is the radian measure of the angle defined by these positions?

3. Suppose the angle defined by the bug's initial position and its new position has radian measure 3.17 `rad`. What is the length of the arc the bug traveled through? Assume the fan has a two-foot radius.

4. Jade turns up the speed of the fan so that its blades make one complete rotation every second. What is the angular speed for the fan?

5. Suppose the angular speed of the fan is $\omega = 1.17$ `rad` per second. What is the length of the arc the bug travels through in two seconds? Assume the fan has a two-foot radius.

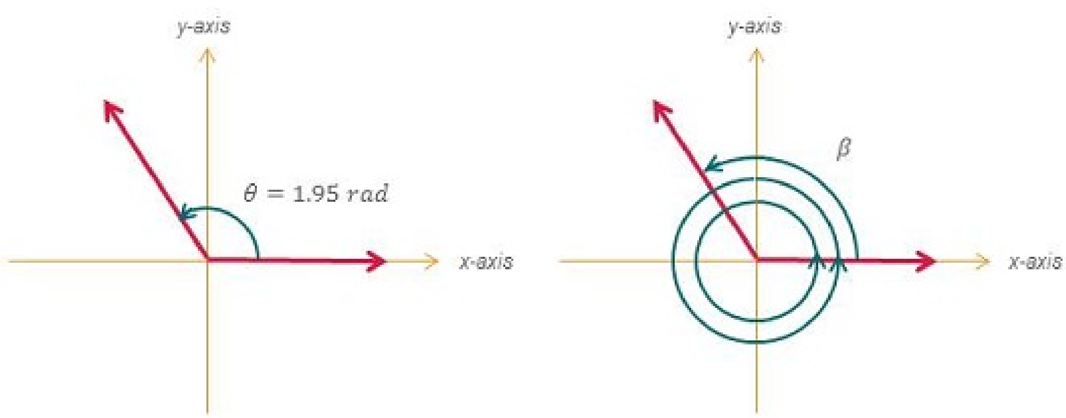
6. What would be a *clockwise-oriented* measure for angles coterminal to those whose radian measure is given below?

   (a) $\theta = \frac{4\pi}{3}$ `rad` (b) $\theta = \frac{8\pi}{5}$ `rad` (c) $\theta = 2.45$ `rad`

7. What would be a *counterclockwise-oriented* measure for angles coterminal to those whose radian measure is given below?

   (a) $\theta = -\frac{4\pi}{3}$ `rad` (b) $\theta = -\frac{\pi}{2}$ `rad` (c) $\theta = -1.73$ `rad`

8. The diagram below shows the radian measures for two coterminal rotation angles. If $\theta = 1.95$ `rad`, what is the value of $\beta$? (The value of $\beta$ must measure the complete rotations shown.)




9. Suppose a rotation angle has measure $\theta = 1.88$ `rad`. An angle coterminal to this one is obtained by adding seven complete clockwise-oriented rotations to the original angle. What is the radian measure $\beta$ of this new angle?

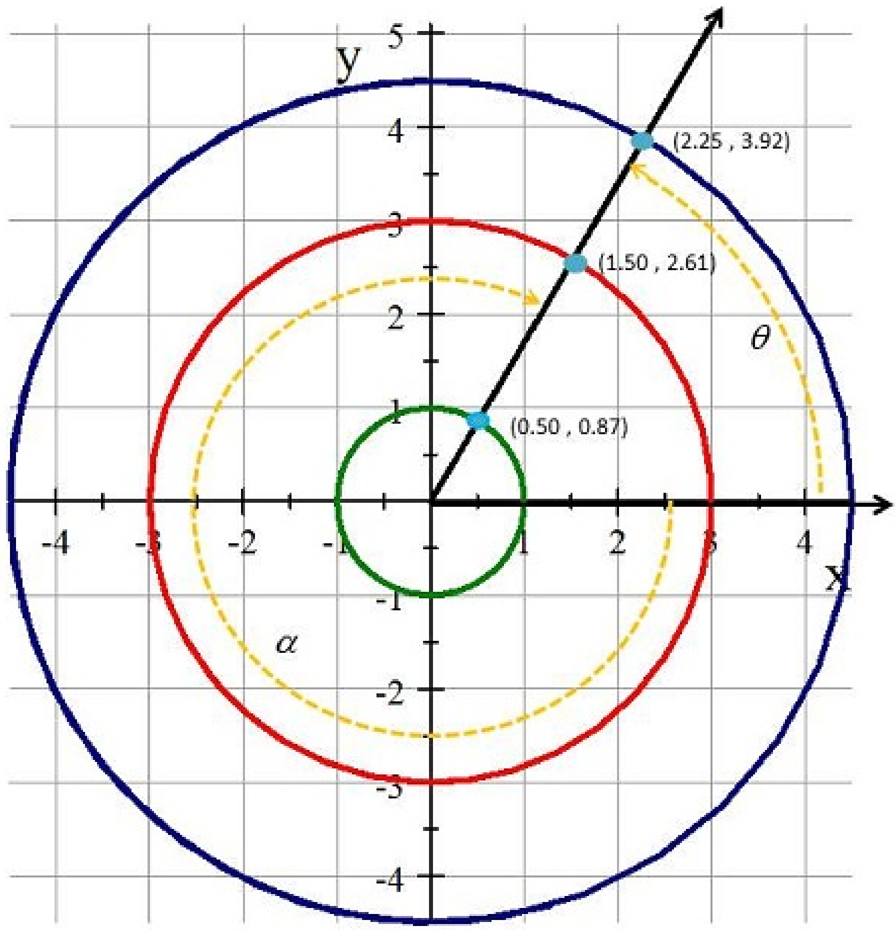
10. Determine the measure of the fundamental rotation angle associated with each angle whose radian measure is given below. By how many complete rotations (and in what direction) do these measures differ from the fundamental measure?

(a) $\beta = \dfrac{23\pi}{7}\ \texttt{rad}$ (b) $\alpha = -\dfrac{33\pi}{5}\ \texttt{rad}$ (c) $\delta = -28.91\ \texttt{rad}$

## 3 Investigation 3 — The Trigonometric Functions

In the last two investigations, we introduced radian measure for angles and explored some of the notation and conventions we use when reasoning with rotation angles. In this investigation, we will introduce three very important functions associated with the measure of angles.

The diagram below shows two coterminal rotation angles in standard position, one with radian measure $\theta$, and the other with radian measure $\alpha$. It also shows the points where the terminal side of this angle intersects three different circles centered at the origin.




**1.** The *tangent function* takes as input the radian measure of an angle in standard position and produces the slope of its terminal ray as output. If $u$ is the radian measure of the angle, and $m$ is the slope of its terminal ray, then we let $m = \tan(u)$ represent this relationship. What is the value of $\tan(\theta)$ in the diagram above? What is the value of $\tan(\alpha)$ in the diagram above?

**2.** How would the values of $\tan(\theta)$ and $\tan(\theta + 2\pi)$ compare? Explain the reasoning behind your answer.

**3.** Consider the value of the $x$-coordinate for each of the three points shown above. In each case, what *percentage of the radius* of the circle is this value (written as a decimal) ? Your answer will only be accurate to the nearest hundredth. What do you notice?

**4.** Consider the value of the $y$-coordinate for each of the three points shown above. In each case, what *percentage of the radius* of the circle is this value (written as a decimal) ? (Your answer will only be accurate to the nearest hundredth.) What do you notice?

Suppose that an angle in standard position has radian measure $u$ and let $P = (x, y)$ be the point where the terminal ray of this angle intersects the circle of radius $R$ centered at the origin. The *cosine function* takes the radian measure $u$ as its input and produces as its output *the percentage of* $R$ left or right of the $y$-axis we must move to reach the $x$-coordinate of $P$. (The precentage is negative if we have to move left and positive if we have to move right.) Written as a decimal, this precentage is simply the value of $x$ divided by the value of $R$.

We use the special symbol "cos" as the name of this function. In symbols, we have

$$\cos(u) = \frac{x}{R}$$

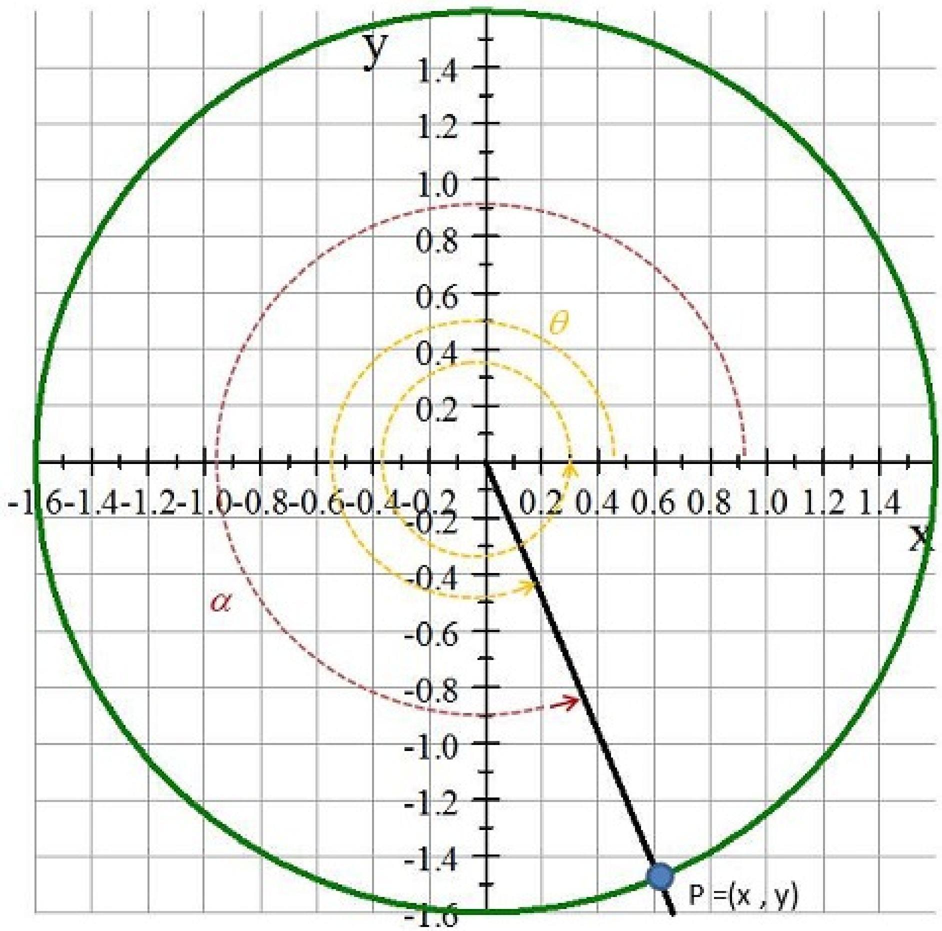
The *sine function* takes the radian measure $u$ as its input and produces as its output *the percentage of* $R$ above or below the $x$-axis we must move to reach the $y$-coordinate of $P$. (The precentage is negative if we have to move below and positive if we have to move above.) Written as a decimal, this precentage is simply the value of $y$ divided by the value of $R$.

We use the special symbol "sin" as the name of this function. In symbols, we have

$$\sin(u) = \frac{y}{R}$$

**5.** In the diagram above, what is the value of $\sin(\theta)$ and $\sin(\alpha)$? What is the value of $\cos(\theta)$ and $\cos(\alpha)$?

**6.** The diagram below shows two coterminal angles in standard position with radian measure $\theta$ and $\alpha$.




**Part (a)** Estimate the coordinates of the point $P$.

**Part (b)** Use these estimates to approximate the value of $\sin(\theta)$ and $\cos(\theta)$ and the values of $\sin(\alpha)$ and $\cos(\alpha)$.

**Part (c)** What is the slope of the terminal ray for these coterminal angles? What is the approximate value of $\tan(\theta)$ and $\tan(\alpha)$?

**7.** Suppose two coterminal angles in standard position have radian measures $u$ and $v$, respectively. How would $\sin(u)$ and $\sin(v)$ compare? How would $\cos(u)$ and $\cos(v)$ compare? Explain your reasoning.

**8.** Mavis starts skiing around a circluar trail whose radius is four kilometers. Assume the center of the track lies at the origin of a grid, and suppose her starting point is on the positive $x$-axis. (This means her starting point is $(4, 0)$.) She decides to stop and take a break at the point $(-2.68, 2.97)$. What percentage of the radius of the track has she moved left or right of the $y$-axis? What percentage of the radius has she moved above or below the $x$-axis?

**9.** Imagine a ray extending from the center of the track to the point where Mavis is resting. This serves as the terminal ray for an angle in standard position. If $u$ is any radian measure for this angle, what is the value of $\cos(u)$? What is the value of $\sin(u)$? What is the value of $\tan(u)$?

**10.** Suppose that $P = (x, y)$ is a point on a circle of radius $R$ centered at the origin. Explain why it is not possible for the value of $x$ or the value of $y$ to be more than 100% of the value of $R$.

**11.** Consider a circle of radius $R$ centered at the origin. Find a point $P$ on the circle whose $x$-coordinate is 100% of $R$ right of the $y$-axis. Find a point $Q$ on the circle whose $y$-coordinate is 100% of $R$ left of the $x$-axis.

**Supplemental Problems.**

1. Suppose that $P = (8, -3)$ lies on the terminal ray of an angle in standard position. If $\theta$ is the radian measure of this angle, determine the values of $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$.

2. Suppose that $P = (-2, 4)$ lies on the terminal ray of an angle in standard position. If $\theta$ is the radian measure of this angle, determine the values of $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$.

3. Explain why the following statement is true — "If $\alpha$ and $\beta$ are radian measures for two coterminal angles in standard position, then $\sin(\alpha) = \sin(\beta)$ and $\cos(\alpha) = \cos(\beta)$.

4. If $\alpha$ and $\beta$ are radian measures for two coterminal angles in standard position, then is it true that $\tan(\alpha) = \tan(\beta)$? Think carefully before answering.

5. Explain why the following statement is true — "If $\theta$ is the radian measure for an angle in standard position, then $\sin(\theta) = \sin(\theta \pm 2\pi)$ and $\cos(\theta) = \cos(\theta \pm 2\pi)$.

6. The *reciprocals* of the three trigonometric functions appear often enough in mathematical formulas that they are also given special names. Let $\theta$ be the radian measure of an angle in standard position.

   *Secant* of $\theta$ is defined by $\sec(\theta) = \dfrac{1}{\cos(\theta)}$     *Cosecant* of $\theta$ is defined by $\csc(\theta) = \dfrac{1}{\sin(\theta)}$

   *Cotangent* of $\theta$ is defined by $\cot(\theta) = \dfrac{1}{\tan(\theta)}$

   (a) Suppose that $P = (4, -3)$ lies on the terminal ray of an angle in standard position. If $\theta$ is the radian measure of this angle, determine the values of $\sec(\theta)$, $\csc(\theta)$, and $\cot(\theta)$.

   (b) Let $\theta$ be the radian measure of an angle in standard position. Suppose we know that this angle cuts an arc exactly half the circumference of a circle having a four foot radius. What is the value of $\sec(\theta)$, $\csc(\theta)$, and $\cot(\theta)$?

7. Draw a circle whose center lies at the origin of a rectangular grid. You may choose the radius to be whatever you like. Let $x$ represent values on the horizontal axis and let $y$ represent values on the vertical axis of this grid.

   (a) Plot the four points where your circle intersects the $x$ and $y$ axes along with their coordinates.

   (b) Suppose $a, b, c, d$ are the radian measures of the angles in standard position whose terminal rays lie on the $x$ and $y$ axes. What are the radian measures of these angles?

   (c) Which of these angle measures produces an output of 0 from the cosine function?

   (d) Which of these angle measures produces an output of 0 from the sine function?

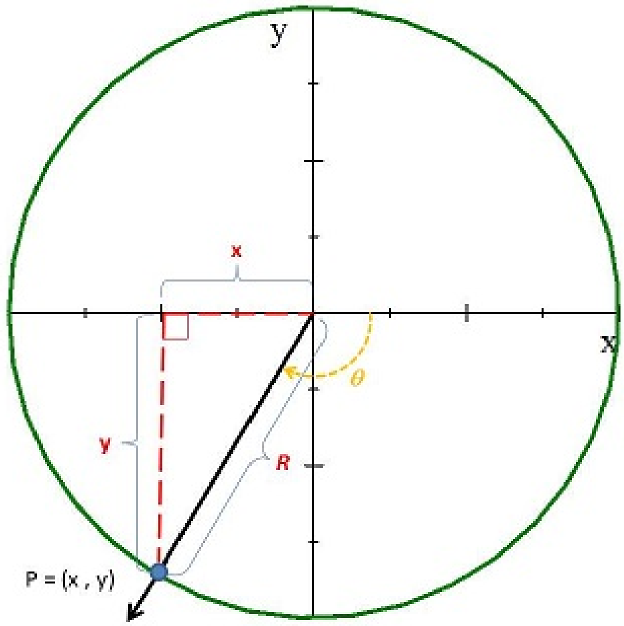
   (e) Are there any other angle measures that will produce an output of 0 for the sine or cosine function? Think carefully and explain your reasoning.

8. Use your answers to Problem 7 to determine *all* angle measures for which the secant function is undefined.

9. Use your answers to Problem 7 to determine *all* angle measures for which the cosecant function is undefined.

# 4 Investigation 4 — Relationships Between the Trigonometric Functions

In the last investigation, we introduced the sine, cosine, and tangent functions. In this investigation, we will introduce a few of the way these functions relate to one another. The diagram below shows an angle in standard position with radian measure $\theta$ along with a circle of radius $R$ centered at the origin. Consider the point $P = (x, y)$ where the terminal side of the angle intersects the circle.




If we drop a vertical line segment down from the point $P$, this line intersects the $x$-axis at a right angle. The length of this segment is $|y|$, while the length of the horizontal line segment between the origin and this intersection point has length $|x|$. These segments form the legs of a right triangle whose hypotenuse has length $R$. Now, since $|y|^2 = y^2$, the Pythagorean Theorem tells us

$$R^2 = x^2 + y^2$$

**1.** Suppose an angle is in standard position and has radian measure $\theta$. Suppose the point $P = (-2, 3)$ is on the terminal ray of the angle.

**Part (a)** What is the exact distance $R$ between $P$ and the origin? (Do not give a decimal approximation.)

**Part (b)** What are the exact values of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$?

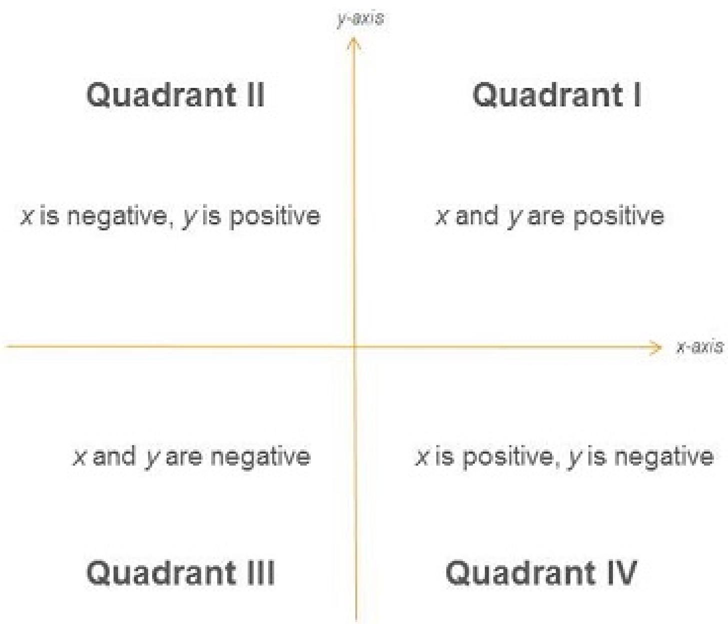
**2.** How can we rewrite the equation $R^2 = x^2 + y^2$ to obtain the equivalent equation $1 = [\cos(\theta)]^2 + [\sin(\theta)]^2$?

When working with trigonometric functions, it is common to write $\sin^2(\theta)$ in place of $[\sin(\theta)]^2$. It is common to do this for the cosine and tangent functions as well. Adopting this convention, we have the so-called *Pythagorean Identity* relating the outputs of the sine and cosine functions — for any angle in standard position with radian measure $\theta$,

$$1 = \cos^2(\theta) + \sin^2(\theta)$$

Notice that the Pythagorean Identity can be rewritten as either $\cos^2(\theta) = 1 - \sin^2(\theta)$ or $\sin^2(\theta) = 1 - \cos^2(\theta)$. Consequently, if we happen to know the value of either $\sin(\theta)$ or the value of $\cos(\theta)$, we can determine the value of the other *at least up to a plus or minus sign.*

A rectangular grid naturally divides the plane into four quadrants. It is customary to name these quadrants in the counterclockise direction as shown in the diagram below.

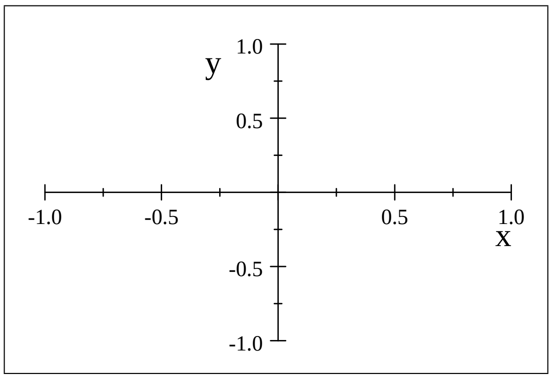


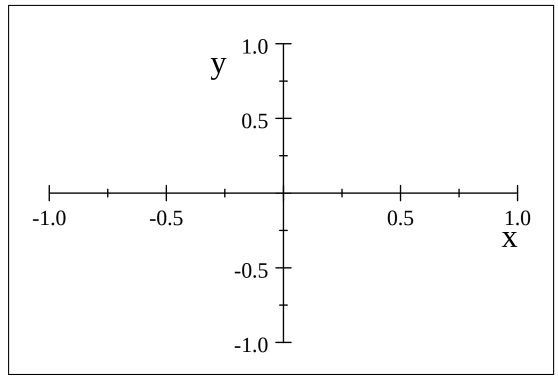

If an angle is in standard position, we say that angle *lies in a quadrant* provided it's terminal ray is in that quadrant. For example, if an angle in standard position has the point $P = (-3, 1)$ on its terminal ray, then this angle lies in Quadrant II. We know this because this is the quadrant where the first coordinate of a point is negative and the second coordinate is positive.

**3.** Suppose an angle in standard position lies in Quadrant III. If $\theta$ is any radian measure of this angle and $\sin(\theta) = -\frac{1}{4}$, use the Pythagorean Identity and the quadrant information to determine the value of $\cos(\theta)$.

**4.** Suppose an angle is in standard position and has radian measure $\theta$. If the terminal ray of this angle is vertical, why is there a problem defining the tangent of $\theta$?

**5.** On each of the grids below, draw a different angle in standard position whose terminal ray is vertical. What is the radian measure of each angle you made, and how do you know?




**6.** Suppose an angle is in standard position and suppose $\theta$ is the radian measure for this angle. As long as the terminal ray of the angle is not vertical, use the definitions of the sine and cosine functions and some algebra to explain why

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

**7.** Suppose that an angle in standard positive has radian measure $\theta$. Furthermore, suppose we know the angle lies in Quadrant II and $\tan(\theta) = -\frac{3}{5}$.

**Part (a)** Millwood says that this information implies $\sin(\theta) = 3$ and $\cos(\theta) = -5$. Explain why Problem 2 makes this impossible.

**Part (b)** Find the correct values of $\sin(\theta)$ and $\cos(\theta)$.

**8.** Arnie is skiing in the clockwise direction around a circular track with a 5.5 kilometer radius. Suppose the center of the track is placed at the origin of a rectangular grid so that Arnie's starting point is on the positive $x$-axis. Arnie stops for a rest at a point $P$ on the track.

**Part (a)** If Arnie has traveled 15% of the track radius to the left of the $y$-axis, what is the $x$-coordinate of the point $P$?

**Part (b)** What is the $y$-coordinate of the point $P$?

**Part (c)** Let $\theta$ be any radian measure for the angle in standard position whose terminal side passes through the point $P$. What is the value of $\tan(\theta)$?

**Supplemental Problems.**

1. Let $A$ be an angle in standard position whose radian measure is $\theta$. If the angle lies in Quadrant III and $\cos(\theta) = -\frac{1}{4}$, what is the value of $\sin(\theta)$ and $\tan(\theta)$?

2. Let $A$ be an angle in standard position whose radian measure is $\theta$. If the angle lies in Quadrant II and $\sin(\theta) = \frac{2}{3}$, what is the value of $\sec(\theta)$ and $\cot(\theta)$?

3. Let $\theta$ be the radian measure of any angle in standard position. Use the Pythagorean Identity and the definition of the secant function show that
$$\sec^2(\theta) = 1 + \tan^2(\theta)$$
as long as all expressions are defined. Hint: Divide both sides of the Pythagorean Identity by $\cos^2(\theta)$.

4. Let $A$ be an angle in standard position whose radian measure is $\theta$. If the angle lies in Quadrant IV and $\tan(\theta) = \frac{5}{4}$, what is the value of $\sec(\theta)$ and $\sin(\theta)$?

5. Let $A$ be an angle in standard position whose radian measure is $\theta$. Explain why we always have $-1 \leq \cos(\theta) \leq 1$ and $-1 \leq \sin(\theta) \leq 1$.

6. Let $A$ be an angle in standard position lying in Quadrant IV whose radian measure is $\theta$, and suppose that $\tan(\theta) = -\dfrac{4}{3}$.

   (a) Explain why it is wrong to conclude that $\cos(\theta) = 3$ and $\sin(\theta) = -4$, even though $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.

   (b) Determine the correct values for $\cos(\theta)$ and $\sin(\theta)$.

7. Simplify the following expressions as much as possible. Your final form of each expression should contain only sine and cosine factors.

   (a) $\dfrac{\sin(a)}{\csc(a)} + \dfrac{\cos(a)}{\sec(a)}$ (b) $\dfrac{\tan(b)\cot(b)}{\csc(b)}$ (c) $\dfrac{\sin(\theta)}{\sec^2(\theta)} + \dfrac{1}{\csc^3(\theta)}$