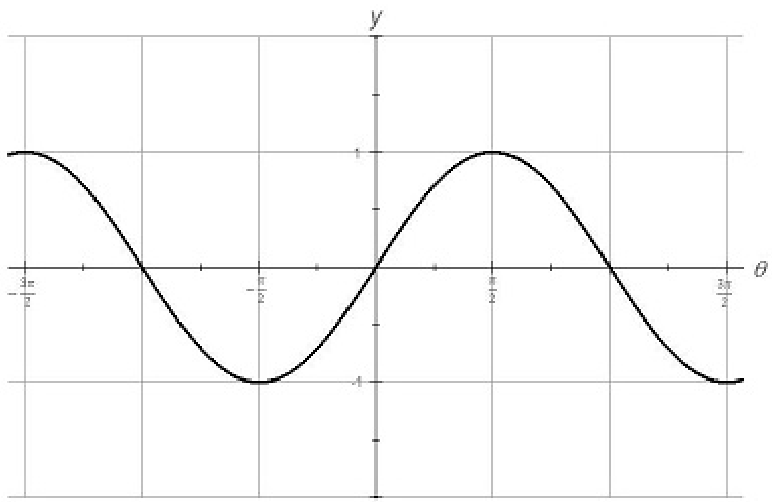
TRIGONOMETRY

1 Investigation 6 — Inverse Trigonometric Functions

There are times when we need to solve for values of the input variable for trigonometric functions. Like the exponential and logarithmic functions, the trigonometric functions are not defined using algebra, so it is not possible to use algebra steps to reverse them. *Unlike* the exponential and logarithmic functions, however, the graphs of the trigonometric functions fail the horizontal line test. This means the sine, cosine, and tangent functions *do not have inverses.*

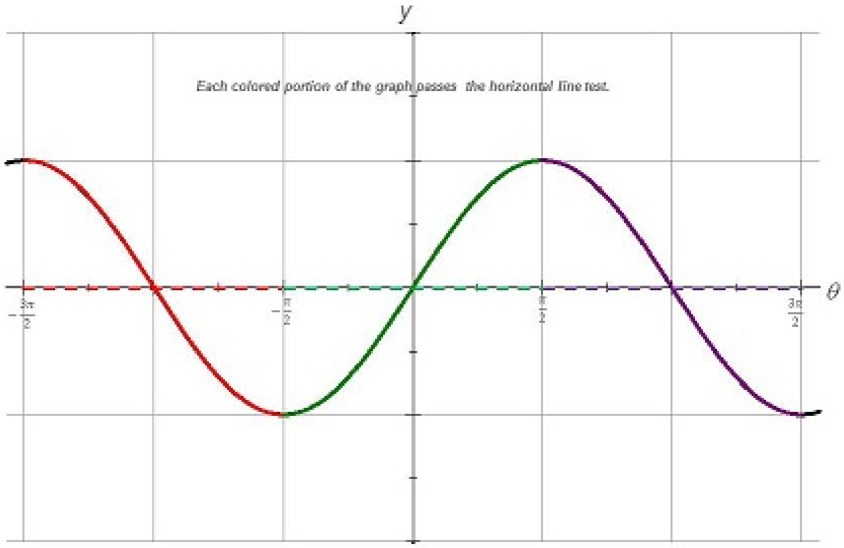


The diagram above shows a portion of the graph of the basic sine function $y = f(\theta) = \sin(\theta)$. As you can see, this function fails the horizontal line test; there are multiple input values associated with each output value. (For example, in the diagram we see the output value $y = 1$ is paired with input values $\theta = -3\pi/2$ and $\theta = \pi/2$.) This means that the process which reverses the sine function does not produce a function. There is no inverse function for the sine function.

We get around this problem by *restricting the domain* of the sine function to a set of input values where the graph does pass the horizontal line test. There are many ways we could do this. However, it is customary to adopt the following strategy when restricting the domain.

- We choose an interval where the output range of the sine function is as large as possible.
- We choose the interval to include as many "practical" angle measures as possible.

The range of the sine function is limited to the interval $-1 \leq y \leq 1$. We could restrict the domain to many different input intervals where the graph covers this range and passes the horizontal line test. Here are the domain restrictions we could make just in the diagram shown.



Of all the possible domain restrictions we could make, we choose to make the restriction $-\pi/2 \leq \theta \leq \pi/2$. This restriction gives us the green portion of the graph shown above. There are many reasons this particular restriction is made, most of which become apparent in calculus. However, there is one simple reason — this restriction includes all of the *acute* angle measures (those angles that can appear in right triangles). This fact makes this particular restriction convenient in applied trigonometry.

THE PRINCIPAL INVERSE SINE FUNCTION

The green portion of the basic sine graph shown above passes the horizontal line test and therefore represents an invertible function. The inverse of this function is called the *principal inverse sine* function or the *arc-sine function.* The principal inverse sine function is traditionally represented by

$$\theta = g(y) = \arcsin(y) \qquad \text{or} \qquad \theta = g(y) = \text{ Sin}^{-1}(y)$$

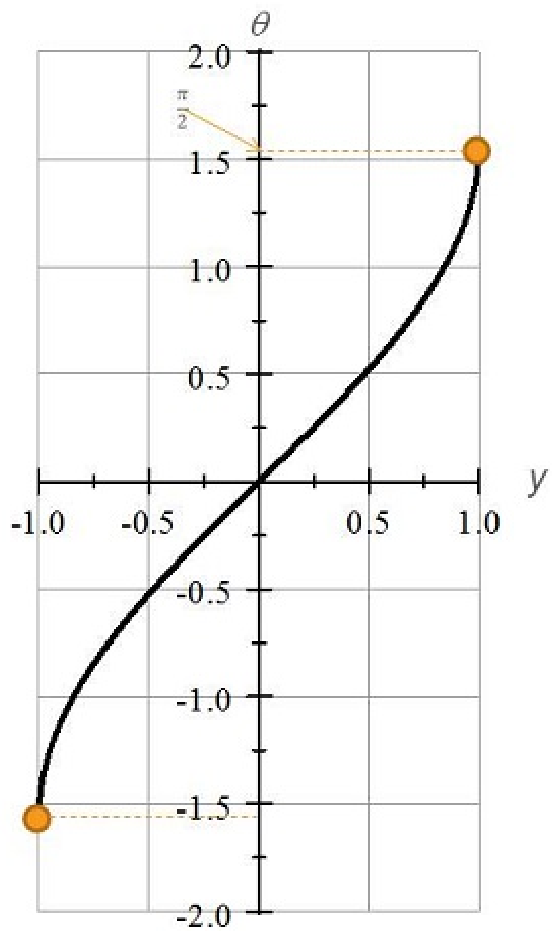
The function g is not a true inverse for the sine function, because the sine function does not have an inverse. The function g serves as a *partial inverse* for the sine function — it only reverses the sine function on the restricted input interval $-\pi/2 \leq \theta \leq \pi/2$.

1. Jerica is skiing around a circular track that has a two kilometer radius. She starts at the point $(2, 0)$ on the track, and stops at the point $(0.34, 1.97)$. Use the arc-sine function and the fact that $1.97 = 2.00\sin(\theta)$ to determine the radian measure θ of the rotation angle Jerica creates as she moves from her starting point to her stopping point. (The arc-sine function probably appears on your calculator as a $\text{Sin}^{-1}()$ key.)

2. The angle whose radian measure is $\alpha = 3.67$ `rad` and the angle whose radian measure is $\beta = -3.99$ `rad` do not lie in the restricted input interval $-\pi/2 \leq \theta \leq \pi/2$. This means that $\arcsin(\sin(\alpha)) \neq \alpha$ and $\arcsin(\sin(\beta)) \neq \beta$. Using the calculator, we know that

$$y = \sin(3.67) \approx -0.504 \qquad y = \sin(-3.99) \approx 0.750$$

The diagram below shows the graph of $\theta = g(y) = \arcsin(y)$. Use this graph to estimate the value of $\arcsin(\sin(\alpha))$ and $\arcsin(\sin(\beta))$.



The previous exercise points out that the arc-sine function is not a true inverse for the sine function. If it were a true inverse, then it would always be true that

$$\arcsin(\sin(\theta)) = \theta$$

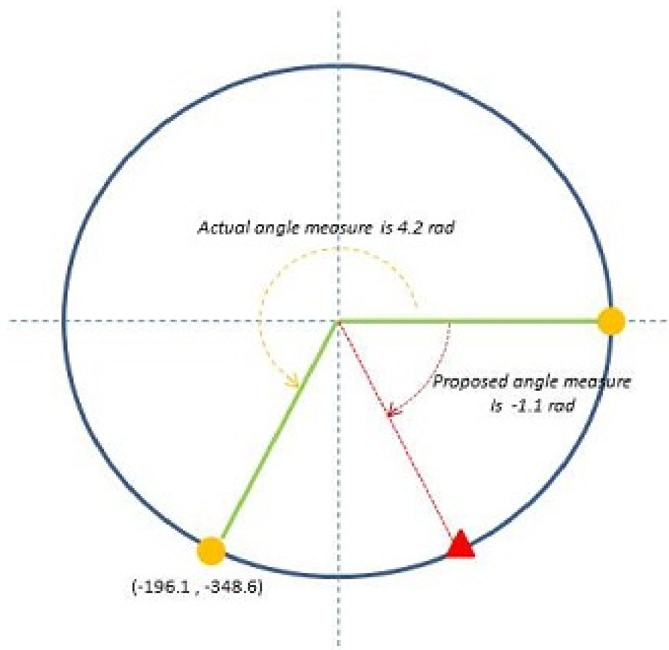
Unfortunately, the equation above holds *only* when we are dealing with the measure of a acute angle in either Quadrant I or Quadrant IV. This limitation can have some important consequences when we are solving problems where we need to determine the measure of an angle. Let's take a look at the issues this limitation can cause.

3. Reggie is running around a circular track that has a radius of 400 feet. Suppose he starts at the point $(400, 0)$ and travels through an arc in the counterclockwise direction having length 1,680 feet before he stops to rest.

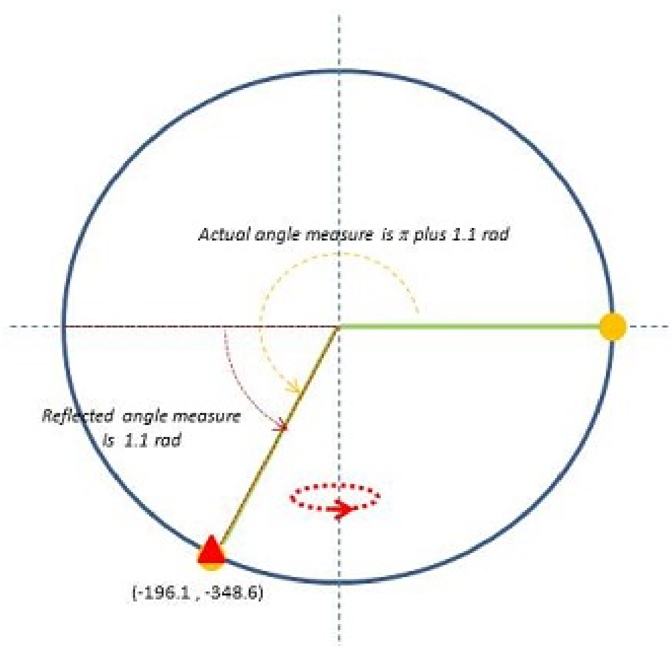
Part (a) What is the radian measure α of the rotation angle Reggie forms as he moves from his starting point to his resting point?

Part (b) Instead of knowing the arc-length Reggie traveled through, suppose we know the *coordinates* of the point where he stopped to rest —Reggie stopped to rest at the point $P = (-196.1, -348.6)$. What is the value for θ that we get if we use the arc-sine function to solve the equation $-348.6 = 400\sin(\theta)$?

We know from Part (a) that the radian measure of the rotation angle is $\alpha = 4.2$ radians, but this is not the answer we get when we solve the equation $-348.6 = 400\sin(\theta)$. What is worse, however, is that the answer we get is not even the measure of an angle *coterminal* to Reggie's true rotation angle.



There is a relationship between the proposed solution $\theta = -1.1$ `rad` and the actual angle measure $\alpha = 4.2$ `rad`, but this relationship is not clear — at least until we *reposition* the angle with measure α. Look what happens if we take the angle with measure θ and *reflect it around the vertical axis.*



We can always use the arc-sine function to solve equations of the form $b = a\sin(\theta)$ for the angle measure θ. Unfortunately, the solution we get will always be the measure of an acute angle in standard position that lies in Quadrant I or Quadrant IV. Based on the context of the particular problem we are trying to solve, *this may not be the solution we are looking for.* To find the solution that satisfies the conditions in our problem, we need extra information — in particular, we need to know what quadrant the desired angle should lie in and what the direction of rotation should be.

SOLVING EQUATIONS USING ARC-SINE

Suppose we want to solve the equation $b = a\sin(\theta)$.

- The expression $\theta = \arcsin\left(\frac{b}{a}\right)$ is always a solution, but θ will be the measure of an acute angle (in standard position) in Quadrant I or Quadrant IV.
- If we need the measure α of a *different* angle that also solves this equation, then we need more information — we need to know the quadrant of the angle and its direction of rotation.
  - If the angle we want lies in Quadrant I or Quadrant IV, then angle of measure α is coterminal with the angle of measure θ.
  - If the angle we want lies in Quadrants II or III, then the angle of measure α is coterminal with the angle of measure $\pi - \theta$.

For example, in Problem 3 Part (b), we are told that Reggie stopped at the point $P = (-196.1, -348.6)$. This point lies on the terminal ray of the angle whose radian measure we want to find. Since this point lies in Quadrant III, and since the solution to the equation $-348.6 = 400\sin(\theta)$ we obtained using the arc-sine function is $\theta \approx -1.1$ `rad`, we know that the angle whose measure we want to find will be coterminal to the angle with measure

$$\alpha = \pi - (-1.1) \approx 4.2 \texttt{ rad}$$

Since the problem also tells us that the angle is measured in the counterclockwise direction, we know α is the measure we seek.

4. Donnelle is riding a ferris wheel, and her distance above the ground in feet is given by the function

$$h = f(t) = 65 + 59\sin\left(\frac{4}{5}t - 3\right)$$

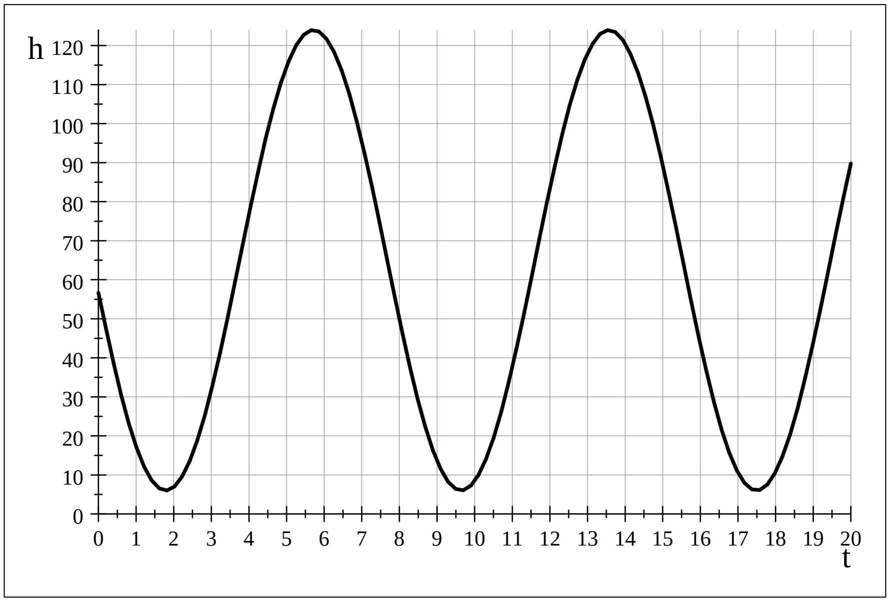
where t is the number of seconds since the ferris wheel started rotating. Use the arc-sine function to find one solution to the equation

$$45 = 65 + 59\sin\left(\frac{4}{5}t - 3\right)$$

5. The diagram below shows a graph of Donnelle's function f from the previous problem.

Part (a) On this graph, mark the approximate ordered pair (t, h) that corresponds to your solution from Problem 2.

Part (b) Use the graph to determine all of the solutions to the equation $42 = 65 + 59\sin\left(\frac{4}{5}t - 3\right)$ that can be found from the diagram.



Part (c) There will be more solutions to this equation for values of t larger than 20 seconds. Based on the pattern you see in your solutions, what will the next two approximate solutions larger than 20 seconds be for the equation?

As Problem 5 points out, there are drawbacks to using the arc-sine function to solve equations that did not show up when we used logarithm functions to solve exponential equations, or when we used exponential functions to solve logarithmic equations. The arc-sine function can only give us *one* solution to a sinusoidal equation, even though that equation will have *infinitely many* solutions.

Example 1 *Use the properties of the arc-sine function to rewrite* $y = h(t) = \sec\left[\arcsin\left(\frac{3t}{4}\right)\right]$ *as an algebraic function of* t.

Solution. A function is *algebraic* provided its formula is constructed using only arithmetic operations. We know that the output of the arc-sine function is the radian measure of an acute angle. Let θ be the measure of this angle. This means

$$\theta = \arcsin\left(\frac{3t}{4}\right) \qquad \text{and} \qquad \sin(\theta) = \frac{3t}{4}$$

Now, if we let $P = (a, b)$ be any point on the terminal side of the angle in standard position whose measure is θ, then we know that

$$\frac{b}{\sqrt{a^2+b^2}} = \sin(\theta) = \frac{3t}{4}$$

Therefore, we can let $b = 3t$, and we know $4 = \sqrt{a^2 + b^2}$. The second equation tells us

$$4 = \sqrt{a^2 + b^2} \Longrightarrow 16 = a^2 + 9t^2 \Longrightarrow \pm\sqrt{16 - 9t^2} = a$$

We can use the properties of the arc-sine function to say more, however. The way we have defined the arc-sine function, the angle whose measure is θ must lie in Quadrant I or Quadrant IV. In either case, *the x-coordinate of the point P must be positive.* Therefore, we know

$$h(t) = \sec\left[\arcsin\left(\frac{3t}{4}\right)\right] = \sec(\theta) = \frac{4}{\sqrt{16 - 9t^2}}$$

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6. Use the properties of the arc-sine function to rewrite $y = h(t) = \tan\left[\arcsin\left(\frac{2}{5t}\right)\right]$ as an algebraic function.

Supplemental Problems.

1. Betula, Anne, and Marques are riding bicycles around a circular track that has a radius of 500 feet. They each start at the point $(500, 0)$ and travel counterclockwise less than one time around the track before stopping.

   (a) If Anne stops at the point $(382.42, 322.11)$, then what is the radian measure of the rotation angle she traveled through from her starting point to her stopping point?

   (b) If Marques stops at the point $(-468.22, -175.39)$, then what is the radian measure of the rotation angle he traveled through from her starting point to his stopping point?

   (c) If Betula stops at the point $(-161.64, 473.15)$, then what is the radian measure of the rotation angle she traveled through from her starting point to her stopping point?

2. Nigel decides to join the cyclists and also starts at the point $(500, 0)$. However, he rides around the track three times in the counterclockwise direction before he goes on to stop at the point $(-314.10, 389.00)$.

What is the radian measure of the rotation angle he traveled through from his starting point to his stopping point?

3. Use the arc-sine function to determine one solution to the equation $43 = 100 + 30\sin\left(\frac{2}{3}x + 5\right)$.

It is possible to define partial inverse functions for all of the trigonometric functions; however, only the partial inverse functions for the sine and tangent functions are commonly used. The *arc-tangent* function is defined in much the same way that the arc-sine function is defined — the domain of $m = f(\theta) = \tan(\theta)$ is restricted to the input interval $-\pi/2 < \theta < \pi/2$ and the inverse function is constructed on this restriction. (Note the strict inequalities — the tangent function is not defined for $\theta = \pm\pi/2$.) The function $\theta = g(m) = \arctan(m)$ gives the radian measure of an acute angle (in standard position) in Quadrant I or Quadrant IV whose terminal ray has slope m. The arctangent function has the same limitations that the arc-sine function has.

4. A ferris wheel has a radius of fifty feet, and its lowest point is ten feet off the ground. The boarding platform for this wheel is located sixty feet above ground. Wilma boards the ferris wheel, and it starts rotating counterclockwise until it malfunctions. When Wilma's gondola stops, the beam connecting it to the center of the wheel has slope $m = 2.2$. Let θ be the radian measure of the rotation angle made as Wilma moved from her starting point to her stopping point. Assume this angle is in standard position, with the boarding platform on the positive x-axis.

   (a) If Wilma was *rising* when she stopped, what quadrant is this angle in?

   (b) Use the arc-tangent function (which will appear as a Tan^{-1} key on your calculator) and the fact that $2.2 = \tan(\theta)$ to determine the radian measure of the angle θ.

   (c) Suppose Fred boarded the same ferris wheel before Wilma, and when Wilma boarded, she was *directly opposite* Fred on the wheel. When the ferris wheel stopped, what was the radian measure of Fred's rotation angle?

(d) Barney boarded the ferris wheel after Fred but before Wilma. When the ferris wheel stopped, Barney was at the point $P = (-25.24, 43.16)$. What is the slope m of the beam connecting Barney's gondola to the center of the wheel?

(e) Use the fact that $m = \tan(\theta)$ and the arc-tangent to determine the measure θ of Barney's rotation angle. Be careful about the quadrant you are in — did you find Barney's angle measure, or the angle measure for a point *directly opposite* Barney's point on the wheel?

5. Rewrite the following functions as algebraic functions of the input variable.

(a) $y = h(t) = \sec\left[\arcsin\left(\frac{2t}{3}\right)\right]$ (b) $y = f(x) = \sin\left[\arctan\left(\frac{1}{x}\right)\right]$