ALGEBRA REVIEW
UNIT 4 --- Working with Quotients

In this unit, we will look carefully at how quotients are manipulated in algebra. We will start by reviewing some of the basic properties of quotients.

Let A and B be expressions. The expression $\frac{A}{B}$ is called the ***quotient*** of A over B.

- The quotient of A over B represents the end result of dividing the expression A by the expression B.

The expression B is called the ***denominator*** of the fraction. The word "denominator" comes from the Latin term for "collection" or "type."

The expression A is called the ***numerator*** of the fraction. The word "numerator" comes from the Latin term for "counting."

The words “numerator” and “denominator” come from the way we understand division of positive integers.

If A and B are positive integers then the expression $\frac{A}{B}$ tells us how we distribute A among B boxes so that each box contains the same amount, and the sum of all the portions is equal to A.

In other words, we “count out” the amount A so that it is equally distributed among B “collections.”

In the context of positive integers, the expression $\frac{A}{0}$ does not make sense, because it is not possible to “count out” the amount A so that it is equally distributed among zero boxes.

This is one reason we do not allow “division by 0” in any context.

When we manipulate quotients of expressions, there are five rules that we must follow. These rules all come from our understanding of what quotients of positive integers mean. Let’s take a look at these rules.

Multiplication Rule for Quotients

MR1 If A and B are expressions, then $\frac{A}{B} = A \cdot \frac{1}{B}$.

MR2 If A , B , C and D are expressions, then $\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$.

Reciprocal Rule for Quotients

RR1 If B is an expression, then $1 = B \cdot \frac{1}{B}$.

RR2 If B is an expression, then $B = \frac{1}{1/B}$.

Addition Rule for Quotients

AR If A , B and C are expressions, then $\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}$.

Addition of quotients is only defined when the denominators are the same.

EXAMPLE 1. Multiply the quotients $\frac{x}{2}$ and $\frac{x-1}{x+1}$. Use the distributive rule to expand the numerator and denominator.

$$\frac{x}{2} \cdot \frac{x-1}{x+1} = \frac{x(x-1)}{2(x+1)} \quad \textit{(Apply MR2)}$$

$$= \frac{x \cdot x + x \cdot (-1)}{2 \cdot x + 2 \cdot 1} \quad \textit{(Apply distributive rule)}$$

$$= \frac{x^2 - x}{2x+2} \quad \textit{(Simplify.)}$$

Exercise 1

Multiply the quotients $\frac{3}{t-2}$ and $\frac{p}{4}$. Use the distributive rule to expand the numerator and denominator.

Exercise 2

Multiply the quotients $\frac{u-3}{x-1}$ and $\frac{x}{u}$. Use the distributive rule to expand the numerator and denominator.

Exercise 3

Multiply the expressions $z - 3z + 1$ and $\frac{z+1}{z}$. Use the distributive rule to expand the numerator.

EXAMPLE 2. Simplify the expression $\frac{x/3}{1/t}$ by rewriting it as a single quotient.

$$\frac{x/3}{1/t} = \frac{x}{3}\cdot\frac{1}{1/t} \quad \textit{(Apply MR1.)}$$

$$= \frac{x}{3}\cdot t \quad \textit{(Apply RR2)}$$

$$= \frac{xt}{3} \quad \textit{(Apply MR1.)}$$

Exercise 4 Simplify the expression $\frac{x/y}{1/v}$ by rewriting it as a single quotient. Show each step in the process and indicate which rule you use in each step.

Exercise 5 Simplify the expression $\frac{3w}{1/(w-5)}$ by rewriting it as a single quotient. Show each step in the process and indicate which rule you use in each step. Use the distributive rule to expand products when applicable.

Exercise 6 Simplify the expression $(h+1)\cdot\frac{3}{1/(h-1)}$ by rewriting it as a single quotient. Show each step in the process and indicate which rule you use in each step. Use the distributive rule to expand products when applicable.

EXAMPLE 3. Show that $\frac{1}{5/p} = \frac{p}{5}$.

$$\frac{1}{5/p} = \frac{1}{5\cdot(1/p)} \quad \textit{(Apply MR1 to the quotient in the denominator.)}$$

$$= \frac{1}{5} \cdot \frac{1}{1/p} \quad \textit{(Apply MR2.)}$$

$$= \frac{1}{5} \cdot p \quad \textit{(Apply RR2)}$$

$$= \frac{p}{5} \quad \textit{(Apply MR1.)}$$

Exercise 7 Follow the procedure in Example 3 to show that $\frac{1}{t/3} = \frac{3}{t}$.

Exercise 8 Follow the procedure in Example 3 to show that $\frac{y}{m/x} = \frac{yx}{m}$.

Exercise 9 Simplify the expression $\frac{x/(y-1)}{y/(x-1)}$ by rewriting it as a single quotient. Use the distributive rule to expand products when applicable.

EXAMPLE 4. Solve the equation $\frac{2x-1}{x+1} = 5$ for x.

$$\frac{2x-1}{x+1} = 5 \Rightarrow (2x-1)\cdot\frac{1}{x+1} = 5$$ *(Apply MR1 to the quotient.)*

$$\Rightarrow (2x-1)\left[\frac{1}{x+1}\cdot(x+1)\right] = 5\cdot(x+1)$$ *(Multiply both sides by $(x+1)$.)*

The goal of the first two steps is to rewrite the equation so that x no longer appears in the denominator.

$$\Rightarrow (2x-1)[1] = 5(x+1)$$ *(Apply RR1.)*

$$\Rightarrow 2x-1 = 5x+5$$ *(Apply the distributive rule.)*

$$\Rightarrow (2x-1)+1 = (5x+5)+1$$ *(Add 1 to both sides.)*

$$\Rightarrow 2x+0 = 5x+6$$ *(Combine like terms.)*

$$\Rightarrow 2x-5x = (5x+6)-5x$$ *(Subtract $5x$ from both sides.)*

$$\Rightarrow -3x = 0+6$$ *(Combine like terms.)*

$$\Rightarrow \frac{-3x}{-3} = \frac{6}{-3}$$ *(Divide both sides by -3.)*

$$\Rightarrow x = -2$$ *(Simplify.)*

Exercise 10 Follow the steps in Example 4 and solve the equation $\frac{x}{3-x} = 10$ for x.

Exercise 11 Follow the steps in Example 4 and solve the equation $\frac{t-2}{3t+1} = 5$ for t.

Exercise 12 Consider the equation $\frac{3}{y-1} = \frac{2}{y+1}$.

Part (a) How would you adjust the first two steps in Example 4 in order to eliminate y from the denominator of each quotient?

Part (b) Solve the equation $\frac{3}{y-1} = \frac{2}{y+1}$ for y.

Exercise 13 Solve the equation $\frac{8}{w} = \frac{6}{2w+5}$ for w.

Exercise 14 Solve the equation $\frac{3h}{4-h} = 2p$ for h. Be sure to exclude any values of p that cause division by 0.

EXAMPLE 5. Rewrite the expression $\frac{2x}{y} - \frac{y}{x-1}$ so that it is a single quotient.

We can add quotients only when their denominators are exactly the same. Therefore, we will need to modify both quotients so that they have the same denominator.

We will let the new "common" denominator be the product of the two denominators, namely $y(x-1)$.

$$\begin{aligned}\frac{2x}{y} &= \frac{2x}{y}\cdot 1\\ &= \frac{2x}{y}\cdot\frac{x-1}{x-1}\\ &= \frac{2x(x-1)}{y(x-1)}\\ &= \frac{2x^2-2x}{yx-y}\end{aligned}$$

Apply RR1.

Apply MR2.

Apply Distributive rule.

$$\begin{aligned}\frac{y}{x-1} &= \frac{y}{x-1}\cdot 1\\ &= \frac{y}{x-1}\cdot\frac{y}{y}\\ &= \frac{yy}{(x-1)y}\\ &= \frac{y^2}{yx-y}\end{aligned}$$

Now that we have transformed each quotient, we may add them together.

$$\begin{aligned}\frac{2x}{y} - \frac{y}{x-1} &= \frac{2x^2-2x}{yx-y} - \frac{y^2}{yx-y}\\ &= \frac{2x^2-2x-y^2}{yx-y}\end{aligned}$$

Exercise 15

Consider the steps shown below. Following Example 5, identify the rule used in each step.

$$\frac{a-b}{a+b} = \frac{a-b}{a+b} \cdot 1$$

$$= \frac{a-b}{a+b} \cdot \frac{x+1}{x+1}$$

RULE USED:

$$= \frac{(a-b)(x+1)}{(a+b)(x+1)}$$

RULE USED:

$$\left.\begin{aligned} &= \frac{a \cdot x + a \cdot 1 + (-b) \cdot x + (-b) \cdot 1}{a \cdot x + a \cdot 1 + b \cdot x + b \cdot 1} \\ &= \frac{ax + a - bx - b}{ax + a + bx + b} \end{aligned}\right\}$$

RULE USED:

Exercise 16

What mistake did Ted make in the following series of steps?

$$\frac{2}{s} = \frac{2}{s} \cdot 1$$

$$= \frac{2}{s} \cdot \frac{1}{t}$$

$$= \frac{2}{st}$$

Exercise 17

What mistake did Marilyn make in the following series of steps?

$$\begin{aligned}\frac{x-1}{t} &= \frac{x-1}{t}\cdot 1\\ &= \frac{x-1}{t}\cdot\frac{y}{y}\\ &= \frac{(x-1)y}{ty}\\ &= \frac{xy-1}{ty}\end{aligned}$$

Exercise 18

Following the steps in Example 5, simplify the expression $\frac{u}{7}+\frac{9}{u}$ by rewriting it as a single quotient.

Exercise 19

Following the steps in Example 5, simplify the expression $\frac{2g}{g-1}-\frac{9}{5}$ by rewriting it as a single quotient.

Exercise 20

Following the steps in Example 5, simplify the expression $\frac{2}{rt}+r$ by rewriting it as a single quotient.

Exercise 21 Consider the equation $4 - \frac{2}{h} = 3$.

Part (a) Simplify the expression on the left-hand side by rewriting it as a single quotient.

Part (b) Using Part (a), solve the equation $4 - \frac{2}{h} = 3$.

Exercise 22 Solve the equation $\frac{t}{r} - \frac{1}{t} = 3$ for r. Be sure to exclude any values of t that cause division by 0.

Exercise 23 Solve the equation $\frac{p-2}{x-1} - \frac{1}{2x} = 2$ for x. Be sure to exclude any values of p that cause division by 0.