ALGEBRA REVIEW
UNIT 5 --- Common Factors

In this unit, we look at the distributive rule from a different perspective.

Suppose A is a sum of expressions. We say that an expression B is a ***common factor*** of A provided all the terms in A are like terms with respect to B.

- If we combine all the like terms with respect to B , we say that we have ***factored B out of the terms***.

EXAMPLE 1. All of the terms in the sum $3x + 3px + 12tx^2$ are like terms with respect to the expression $3x$. If we combine these like terms, we have

$$3x + 3px + 12tx^2 = 1 \cdot (3x) + p \cdot (3x) + 4tx \cdot (3x)$$
$$= (1 + p + 4tx) \cdot (3x)$$

By combining the like terms, we have factored $3x$ out of the expression.

Exercise 1

The terms in the sum $4t - 4$ are like terms with respect to the expression 4. Follow Example 1 to factor 4 out of this sum.

Exercise 2

The terms in the sum $12y - 6y^2$ are like terms with respect to the expression $6y$. Follow Example 1 to factor $6y$ out of this sum.

Exercise 3

The terms in the sum $2(t-1) - 3p(t-1)^3 + (4t-4)$ are all like terms with respect to the expression $(t-1)$. Follow Example 1 to factor $(t-1)$ out of this sum.

Exercise 4

Rosario believes she can factor $5p$ out of the sum $5p + 25p^3 - 5$. When she combines like terms, she comes up with the equation

$$5p + 25p^3 - 5t = (1 + 5p^2 - 5t)(5p)$$

Is Rosario correct? Explain your reasoning.

When we factor an expression out of a sum, we are actually *running the distributive rule backward*.

Applying distributive rule →

$$(1 + p + 4tx)\cdot(3x) \quad = \quad 1\cdot(3x) + p\cdot(3x) + 4tx\cdot(3x) \quad = \quad 3x + 3px + 12tx^2$$

← *Factoring $3x$ out of the sum*

Exercise 5 Apply the distributive rule to the right-hand side of each equation to see if the proposed factoring is valid.

Part (a) $4pt^2 - 12xpt + 4pt \stackrel{?}{=} (t - 3x + 1)(4pt)$

Part (b) $4x - 5y + 8xyt \stackrel{?}{=} (1 - 5y + 2yt)(4x)$

Part (c) $\frac{t}{px} - \frac{2r}{x} \stackrel{?}{=} \left(\frac{t}{p} - 2r\right) \cdot \frac{1}{x}$

EXAMPLE 2. Consider the quotient $\frac{3xy^2-4y}{xy}$.

Notice that y is a factor of the denominator and a common factor of the numerator. This fact will allow us to simplify the quotient.

$$\frac{3xy^2-4y}{xy} = \frac{(3xy-4)y}{xy}$$ *(Factor out y from the numerator.)*

$$= \frac{3xy-4}{x} \cdot \frac{y}{y}$$ *(Apply MR2.)*

$$= \frac{3xy-4}{x} \cdot 1 \quad \text{(As long as } y \neq 0.\text{)}$$ *(Apply RR1 and exclude $y = 0$.)*

$$= \frac{3xy-4}{x} \quad \text{(As long as } y \neq 0.\text{)}$$

In the previous example, we had to exclude $y = 0$ because the expression

$$\frac{3xy - 4}{x}$$

is defined when $y = 0$, but the expression

$$\frac{3xy^2 - 4y}{xy}$$

is NOT defined when $y = 0$. These two expressions will be equal for all values of y EXCEPT for $y = 0$. Note that *both* expressions are undefined when $x = 0$, so we don't have to exclude that possibility. (It is alright to do so, however.)

It is important to note that

$$\frac{3xy - 4}{x} \neq 3y - 4$$

for every nonzero value of x and y. For example, if we let $x = 2$ and $y = 3$, we have

$$\frac{3(2)(3) - 4}{2} = 7 \quad BUT \qquad 3(3) - 4 = 5$$

Exercise 6

Use the value $t = 6$ to show that

$$\frac{t}{t+1} \neq \frac{1}{1+1}$$

Exercise 7 Simplify the quotient $\frac{8t}{24y-8}$ by following Example 2.

Exercise 8 Simplify the quotient $\frac{u^2}{6u^4-u^2}$ by following Example 2. You will need to exclude a value of u.

Exercise 9 Simplify the quotient $\frac{6pt-18py}{12p-18py}$ by following Example 2. You will need to exclude a value of p.

Exercise 10 Simplify the quotient $\frac{3(t-2)^2}{8t-16}$. You will need to exclude a value of t.

Exercise 11 Simplify the quotient $\frac{21x^3+3x}{12x^2}$. Explain why you do not have to exclude any values of x.

Exercise 12 Use a specific value of r to show that the "simplification" below is not valid.

$$\frac{3r^2-7}{r+5} \stackrel{?}{=} \frac{3r+7}{1+5}$$

It is not always obvious that one expression is a factor of another. You are already familiar with many such examples, although you may not be aware of it.

Examples of "hidden" factors in expressions

- $4x^2 - 9 = (2x-3)\cdot(2x+3)$ — Both $2x-3$ and $2x+3$ are hidden factors of $4x^2-9$
- $15t^2 + 32t - 7 = (5t-1)\cdot(3t+7)$ — Both $5t-1$ and $3t+7$ are hidden factors of $15t^2+32t-7$
- $z^3 - y^3 = (z-y)\cdot(z^2+zy+y^2)$ — Both $z-y$ and z^2+zy+y^2 are hidden factors of z^3-y^3

The equations above all represent "factored polynomials," and you probably spent a lot of time in high school learning techniques for this kind of factoring.

An expression is a ***polynomial*** provided it is a sum of terms ax^n where a is a real number and n is an integer that is not negative.

- The largest value of n appearing in the expression is called the ***degree*** of the polynomial.

In practice, you seldom need to factor a polynomial "from scratch" as the techniques you learned show you how to do. More often, you need to determine whether a given expression serves as a factor for a polynomial.

Determining whether a given expression is a factor of a polynomial is actually fairly easy. The method of "long division" provides the way.

EXAMPLE 3. Use long division to show that $3t + 7$ is a factor of $15t^2 + 32t - 7$.

STEP 1A. Ask yourself "***What should I multiply*** $3t$ ***by in order to obtain*** $15t^2$?

ANSWER: I need to multiply $3t$ **by** $5t$**.**

STEP 1B. Multiply the *entire* expression $3t + 7$ by $5t$ and then *subtract* the result from $15t^2 + 32t - 7$

$$\begin{array}{r r l} & 5t & \leftarrow \textit{Your answer to STEP 1A.} \\ 3t+7 \;) & \overline{15t^2 + 32t - 7} & \\ & \underline{-(15t^2 + 35t)} & \leftarrow \textit{Subtract } 5t \cdot (3t+7) \textit{ from } 15t^2 + 32t - 7 \\ & -3t - 7 & \end{array}$$

EXAMPLE 3 Continued

STEP 2A. Ask yourself "***What should I multiply $3t$ by in order to obtain $-3t$*** ?

ANSWER: I need to multiply $3t$ by -1.

STEP 2B. Multiply the *entire* expression $3t + 7$ by -1 and then *subtract* the result from $-3t - 7$

$$
\begin{array}{r l}
 & \quad\quad 5t - 7 \quad \leftarrow \textit{Your answer to STEP 2A.} \\
3t + 7 \;) & \overline{15t^2 + 32t - 7} \\
 & \underline{-(15t^2 + 35t)} \\
 & \quad\quad -3t - 7 \\
 & \quad\quad \underline{-(-3t - 7)} \leftarrow \textit{Subtract } (-1)\cdot(3t+7) \textit{ from } -3t-7 \\
 & \quad\quad\quad\quad 0
\end{array}
$$

Since the process ends with 0, we know that $3t + 7$ is indeed a factor of $15t^2 + 32t - 7$; in fact, the process tells us that $15t^2 + 32t - 7 = (3t + 7) \cdot (5t - 7)$.

Exercise 13 Use long division to show that $3p+8$ is a factor of $12p^2+17p-40$.

Exercise 14 Use long division to show that $m-3$ is a factor of m^3-3m^2+m-3.

Exercise 15 Use long division to show that $4x+1$ is a factor of $4x^4+x^3+4x+1$.

Exercise 16 Use long division to show that $2y^2+1$ is a factor of

$$2y^4+4y^3+7y^2+2y+3.$$

Long division can also tell you when a given expression is NOT a factor of a polynomial. During the process, if you obtain a polynomial whose degree is SMALLER than the degree of the expression you are considering, the process stops. The expression you are considering is NOT a factor of the polynomial.

EXAMPLE 4. Show that $b + 7$ is NOT a factor of $b^3 - 7b + 6$.

$$
\begin{array}{rl}
 & b^2 - 7b + 42 \\
b + 7 \;) & \overline{b^3 + 0b^2 - 7b + 6} \\
 & \underline{-(b^3 + 7b^2)} \\
 & -7b^2 - 7b + 6 \\
 & \underline{-(-7b^2 - 49b)} \\
 & 42b + 6 \\
 & \underline{-(42b + 294)} \\
 & -288
\end{array}
$$

This polynomial is called the quotient. ($b^2 - 7b + 42$)

Steps 1A and 1B (subtracting $b^3 + 7b^2$ to get $-7b^2 - 7b + 6$)

Steps 2A and 2B (subtracting $-7b^2 - 49b$ to get $42b + 6$)

Steps 3A and 3B (subtracting $42b + 294$ to get -288)

In Step 3B, we obtain -288. This polynomial has Degree 0 , while $b + 7$ has Degree 1. Therefore, the process of long division stops; and we may conclude that $b + 7$ is NOT a factor of $b^3 - 7b + 6$. The last polynomial obtained in the long division process is called the ***remainder***.

The process actually tells us more than this. We now know that

$$b^3 - 7b + 6 \quad = \quad (b + 7) \cdot (b^2 - 7b + 42) - 288$$

Exercise 17

Show that $2r-1$ is NOT a factor of $12r^2-4$. Use your work to write $12r^2-4$ as the polynomial $2r-1$ times a quotient plus a remainder like in Example 4.

Exercise 18

Show that t^2+3 is NOT a factor of t^3+2t+8. Use your work to write t^3+2t+8 as the polynomial t^2+3 times a quotient plus a remainder.

Exercise 19

Consider the quotient $\frac{2a^2-7a+6}{4a^2-13a+10}$.

Part (a)

Use long division to show that $a-2$ is a factor of the numerator.

Part (b)

Use long division to show that $a-2$ is a factor of the denominator.

Part (c)

Use your information from Parts (a) and (b) to simplify the quotient. You will need to exclude a value of a.

Exercise 20

Show that $3x+4$ is a factor of the numerator and the denominator in the quotient below, then use this information to simplify the quotient.

$$\frac{6x^4+8x^3+3x^2+7x+4}{3x^3+4x^2-3x-4}$$