ALGEBRA REVIEW
UNIT 7 --- Working with Negative Powers

In the last unit, we worked with ARP (Additive Rule for Powers), which captures a pattern we observed when we multiply the same expression raised to two different positive integer powers. In this unit, we will introduce negative integer powers. These powers do not correspond directly to multiplication like positive integer powers do.

If A is any expression, how should we *define* the power A^0?

Whatever we define A^0 to be, we want ARP to remain valid. In other words, for any positive integer n, we want

$$A^0 \cdot A^n = A^{0+n}$$

However, since $0 + n = n$, this tells us that

$$A^0 \cdot A^n = A^n$$

Now, *as long as A is nonzero*, we can solve this equation for A^0.

$$A^0 \cdot A^n = A^n \implies \frac{A^0 \cdot A^n}{A^n} = \frac{A^n}{A^n} \quad \textit{(Divide both sides by } A^n\textit{.)}$$

$$\implies A^0 = 1$$

If A is any expression and n is a positive integer, how should we *define* the power A^{-n}?

Whatever we define A^{-n} to be, we want ARP to remain valid. In particular, we want

$$A^n \cdot A^{-n} = A^{n-n}$$

However, since $n - n = 0$, this tells us that

$$A^n \cdot A^{-n} = A^0$$

Now, *as long as A is nonzero*, we know $A^0 = 1$; and we can solve this equation for A^{-n}.

$$A^n \cdot A^{-n} = 1 \implies \frac{A^n \cdot A^{-n}}{A^n} = \frac{1}{A^n}$$ ***(Divide both sides by A^n.)***

$$\implies A^{-n} = \frac{1}{A^n}$$

Let A be any expression and let n be a positive integer. As long as $A \neq 0$, we let $A^0 = 1$ and $A^{-n} = \frac{1}{A^n}$.

- We adopt this definition so that ARP remains valid for the new expressions A^0 and A^{-n} .
- We make no attempt to define 0^0 and 0^{-n}.

EXAMPLE 1. When would the expression $(3x-4)^{-2}$ be undefined?

Since $(3x-4)^{-2} = \frac{1}{(3x-4)^2}$, the expression will be undefined when the denominator is equal to 0.

The square of any nonzero number will be nonzero, so the denominator will be 0 only when $3x - 4 = 0$.

$3x - 4 = 0 \quad \Rightarrow \quad x = \frac{4}{3}$ ***(You can fill in the details for solving this equation.)***

Exercise 1 For what values of t is the expression $(2t+9)^{-3}$ undefined?

Exercise 2 For what values of y and d is the expression $y^{-1} \cdot (4d+7)^{-4}$ undefined?

Exercise 3 When would the equation $(12m+5)^0 = 1$ NOT be valid?

When working with expressions involving negative powers, we usually rewrite them as quotients so that all powers are positive.

Exercise 4 If we rewrite $x^{-1} - (u-5)^{-2}$ as a difference of two quotients, what would the common denominator be?

Exercise 5 Explain why the equation $3t^{-2} = \frac{1}{3t^2}$ is NOT valid. What is the correct way to write this expression as a quotient?

Exercise 6 Rewrite the expression yr^{-2} so that all powers are positive.

EXAMPLE 2. Rewrite $\frac{2yw^{-2}}{5t^{-3}}$ so that all powers are positive.

$$\frac{2yw^{-2}}{5t^{-3}} = \frac{2y}{5}\cdot\frac{w^{-2}}{t^{-3}}$$ *(Separate the expressions raised to negative powers using MR2.)*

$$= \frac{2y}{5}\cdot\frac{{}^{1}\!/_{w^2}}{{}^{1}\!/_{t^3}}$$ *(Rewrite expressions raised to negative powers as quotients.)*

$$= \frac{2y}{5}\cdot\left(\frac{1}{w^2}\cdot t^3\right)$$ *(Apply RR2.)*

$$= \frac{2yt^3}{5w^2}$$ *(Apply MR2.)*

Exercise 7 Rewrite the expression $\frac{4xu^{-4}}{3py^{-2}}$ so that all powers are positive.

Exercise 8 Rewrite the expression $\frac{x^3mt^{-1}}{2yv^2g^{-2}f^{-3}}$ so that all powers are positive.

Exercise 9 Explain why the equation $tx^{-3}+2=\frac{t+2}{x^3}$ is NOT valid. Correctly rewrite the left-hand expression so that all powers are positive.

EXAMPLE 2. Rewrite $tx^{-2} - 3(rp)^{-4}$ as a single quotient in which all powers are positive.

It is important to note that in the expression tx^{-2}, the power -2 applies only to x.

$$tx^{-2} - 3(rp)^{-4} = t \cdot \frac{1}{x^2} - 3 \cdot \frac{1}{(rp)^4}$$ *(Rewrite all expressions so powers are positive.)*

$$= \frac{t}{x^2} - \frac{3}{(rp)^4}$$ *(Apply MR2.)*

$$= \frac{t}{x^2} \cdot 1 - \frac{3}{(rp)^4} \cdot 1$$

$$= \frac{t}{x^2} \cdot \left[\frac{(rp)^4}{(rp)^4}\right] - \frac{3}{(rp)^4} \cdot \left(\frac{x^2}{x^2}\right)$$ *(Apply RR1.)*

$$= \frac{t(rp)^4}{x^2(rp)^4} - \frac{3x^2}{x^2(rp)^4}$$ *(Apply MR2.)*

$$= \frac{t(rp)^4 - 3x^2}{x^2(rp)^4}$$ *(Apply AR.)*

Exercise 10

Rewrite the expression $3w^{-1} + 7$ as a single quotient in which all powers are positive.

Exercise 11

Rewrite the expression $x^{-2} - 4ta^{-3}$ as a single quotient in which all powers are positive.

Exercise 12

Rewrite the expression $\frac{5r^{-2} + 4}{t}$ as a single quotient in which all powers are positive.

Exercise 13

Rewrite the expression $\frac{w^{-1} + 3x}{2d^{-3}}$ as a single quotient in which all powers are positive.

Exercise 14

Solve the equation $4x^{-1} - 3 = x^{-1}$ for x .

Exercise 15

Solve the equation $4t^{-1} - 7 = x^{-1}$ for x . Your final solution should be a quotient in which all powers are positive.

The Additive Power Rule gives us another fact about powers that can help simplify quotients.

If A is any expression, then as long as $A \neq 0$ we have

$$\frac{A^m}{A^n} = A^m \cdot \frac{1}{A^n} = A^m \cdot A^{-n} = A^{m-n}$$

EXAMPLE 4. Simplify the quotient $\frac{3xr^4t^3}{yr^2t^5}$.

$$\frac{3xr^4t^3}{yr^2t^5} = \frac{3x \cdot r^{4-2} \cdot t^{3-5}}{y}$$

(Take the powers of r and t in the denominator and subtract from the corresponding powers in the numerator.)

$$= \frac{3x \cdot r^2 \cdot t^{-2}}{y}$$

$$= \frac{3xr^2}{yt^2}$$

(Rewrite quotient so that all powers are positive.)

Exercise 16 Simplify the quotient $\frac{4ywu^6}{w^2u^3}$.

Exercise 17 Simplify the quotient $\frac{x^{-2}y^3}{tx^3y^{-1}z^2}$.

Exercise 18 Simplify the quotient $\frac{5tr(2p)^4(3p-1)^2}{rp^3(3p-1)^{-2}}$. You will need to exclude values for two variables.