TANGENT LINE APPROXIMATIONS

In this discussion, we will return to what motivated our exploration of derivatives in the first place — the fact that a function $y = f(x)$ is differentiable at a point $(a, f(a))$when there exists a line $y = T_a(x)$ having the following properties:

- The graph of T_a passes through the point $(a, f(a))$.
- The graph of T_a provides an increasingly good approximation to the graph of f as input values for x approach $x = a$.

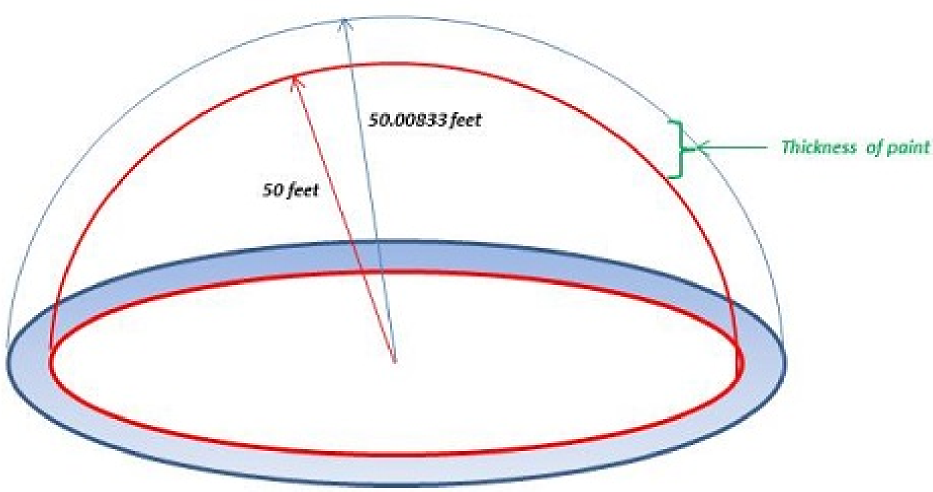
This line, when it exists, is called the tangent line to the graph of f at the point $(a, f(a))$; the slope of this line is called the derivative of f at $x = a$. We have chosen to make tangent lines important simply because it is easier to work with linear functions than it is to work with any other type of function. This was especially true in the days before graphing calculators made it relatively simple to sketch functions.

Example 1 *Suppose that a contractor needs to paint a hemispheric dome that has a radius of fifty feet. Furthermore, suppose that the paint must be applied to a thickness of at most* 0.1 *inches at any point on the dome. What is the volume of paint (in cubic feet) needed to paint the dome?*

Solution. We can approach this problem in two ways — directly by using only the volume formula for a hemisphere, and indirectly using calculus. Let's consider the direct approach first. If we let R represent the radius of a hemisphere, measured in feet, and let V be the volume of the hemisphere, measured in cubic feet, then we know

$$V = f(R) = \frac{2\pi R^3}{3}$$

To tackle this problem directly, we need to consider *two* domes — one having radius 50 feet, and the other having radius 50.00833 feet (where .00833 feet is the maximum allowed thickness of the paint).



The volume of paint required will be the difference in the volumes of these two hemispheres. In symbols,

$$\Delta V = f(50.00833) - f(50) = \frac{2\pi}{3}\left(50.00833^3 - 50^3\right) \approx 130.869 \text{ cubic feet}$$

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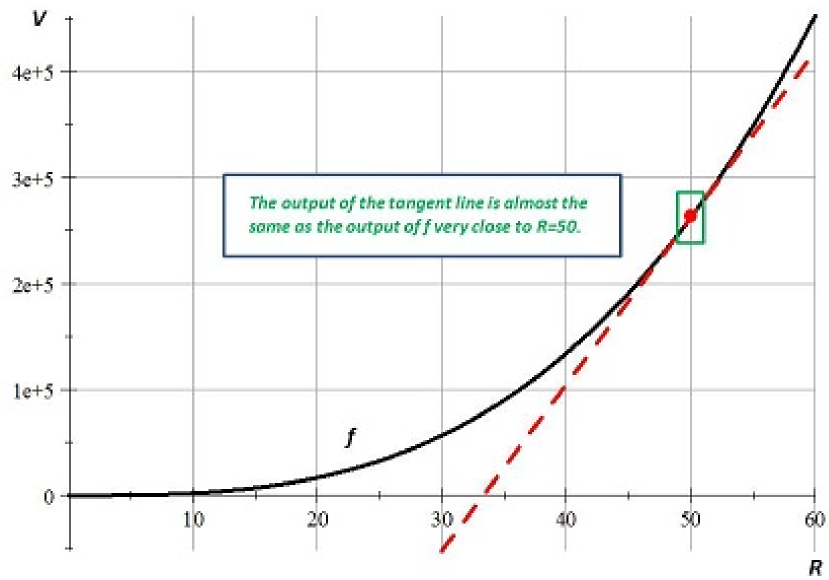
With today's calculator technology, obtaining a numeric solution to this problem is not very difficult. However, a generation ago this would not have been the case. We can take advantage of the differentiablility of the volume function to make approximating the solution easier. To begin, note that

$$f(50) = \frac{2\pi}{3} \cdot (50)^3 = \frac{250000}{3}\pi \text{ cubic feet} \qquad f'(50) = 2\pi \cdot (50)^2 = 5000\pi \text{ cubic feet per radius foot}$$

With this in mind, we can construct the tangent line to the graph of f at the point $(50, f(50))$. This tangent line is

$$V = T_{50}(R) = 5000\pi\,[R - 50] + \frac{250000}{3}\pi$$

Since $R = 50.00833$ feet is very close to $R = 50$ feet, we know that the output value $T_{50}(50.00833)$ will be very close to the output value $f(50.00833)$, as the following diagram shows.



Since we know that $f(50) = T_{50}(50)$, this means we can obtain a very good approximation to ΔV using only the tangent line. Observe that

$$\begin{aligned} \Delta V &\approx T_{50}(50.00833) - T_{50}(50) \\ &= \left(5000\pi\,[50.00833 - 50] + \frac{250000}{3}\pi\right) - \frac{250000}{3}\pi \\ &= 5000\pi\,(.00833) \\ &\approx 130.847 \text{ cubic feet} \end{aligned}$$

The advantage to this approach lies in the fact that, by using the tangent line approximation, we did not have to compute the cube of 50.00833. Of course, by today's calculator standards, this does not seem very important. However, prior to 1980, computations like this required a slide rule and quite a bit of work.

Let's take a closer look at how the two computations unfolded. First, let's consider the direct approach. Estimating the value of ΔV amounted to observing that

$$\Delta V = f(50 + \Delta R) - f(50)$$

On the other hand, approximating ΔV using the tangent line to the graph of f at $(50, f(50))$ amounted to observing that

$$\Delta V \approx T_{50}(50 + \Delta R) - T_{50}(50) = f'(50)\Delta R$$

Now, if we rewrite the last approximation, we have a familiar relationship:

$$\frac{\Delta V}{\Delta R} \approx f'(50)$$

Indeed, we know

$$f'(50) = \lim_{h \longrightarrow 0} \frac{f(50+h) - f(50)}{h} = \lim_{\Delta R \longrightarrow 0} \frac{f(50+\Delta R) - f(50)}{\Delta R} = \lim_{\Delta R \longrightarrow 0} \frac{\Delta V}{\Delta R}$$

Notice that the formula $f'(50)\Delta R$, while only an approximation to the actual change in volume, is *exactly equal* to the change in output for the tangent line. Since we have already used ΔV to represent the actual change in volume, it has become customary to let dV denote the change in the output of the tangent line. Under this custom, we have

$$\Delta V \approx dV = f'(50)\Delta R$$

Furthermore, since the change in the input values of R is the same for both the volume function f and the tangent line, it has become customary to let $dR = \Delta R$ when we are working with the tangent line.

DIFFERENTIAL NOTATION

- Suppose that $y = f(x)$ is a differentiable function. The formula $dy = f'(x)dx$ is called the *differential of y*. In practical terms, it gives the change in the output of the tangent line to the graph of f at the point $(x, f(x))$ for a specified change dx in the input value x. The differential formula is often used to approximate changes in the output of the function f for very small changes in the input value x.

Example 2 *A surveyor needs to compute the area of a plot of land in the shape of an isoceles right triangle. Her instruments have a tolerance (maximum error range) of ± 0.001 feet. If she finds one leg of the triangle to have a length of 400 feet, determine the true error in the computed area of the plot, then use differentials to approximate the error in the computed area of the plot.*

Solution. We know that the area A in square feet of the plot will be given by the formula

$$A = f(x) = \frac{x^2}{2}$$

where x is the length in feet of (both) the base and height of the triangle. The actual error in the area of the plot would be given by

$$\Delta A = f(400.001) - f(400) = \frac{1}{2}\left(400.001^2 - 400^2\right) \approx 0.4000005 \text{ square feet}$$

Now, we know the differential for A is the formula $dA = f'(x)dx$; and since $f'(x) = x$, we know $dA = xdx$. If we let $dx = \Delta x = .001$ feet, then the maximum error ΔA in the area of the plot is approximated by

$$\Delta A \approx dA = 400\,(.001) = 0.4 \text{ square feet}$$

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Example 3 *Suppose the equatorial circumference of a sphere has been measured to be 100 feet, with a tolerance of ± 0.1 foot. Use differentials to estimate maximum computed error and relative error in the volume of the sphere.*

Solution. We need a formula that relates the volume of a sphere to its equatorial circumference. Let V represent the volume, in cubic feet, of the sphere, and let C represent the circumference, in feet, of the sphere. We know that the volume and equatorial circumference are both related to the radius R of the sphere (measured in feet) according to the formulas

$$C = 2\pi R \qquad V = \frac{4\pi R^3}{3}$$

Consequently, we can write V as a function of C according to the formula

$$V = f(C) = \frac{4\pi}{3}\left(\frac{C}{2\pi}\right)^3 = \frac{C^3}{6\pi^2}$$

Now, the differential of this formula will be

$$dV = f'(C) \cdot dC = \frac{C^2}{2\pi^2} \cdot dC$$

To estimate the maximum computed error in the volume, we let $dC = 0.1$ feet, and we let $C = 100$ feet. The maximum computed error in the volume will be

$$\Delta V \approx dV = \left(\frac{100^2}{2\pi^2}\right) \cdot (0.1) \approx 50.661 \text{ cubic feet}$$

This may seem like a lot, but considering the relative error puts things in better perspective:

$$\text{Relative Error:} \quad \frac{\Delta V}{V} \approx \frac{50.661}{5375.256} \approx 0.009$$

The maximum computed error is at most about nine one-thousandths of the computed volume.

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HOMEWORK: Section 3.10, Page 257, Problems 33, 34, 35, 36, and 38