GENERAL TOPOLOGY
An Introduction

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Chapter 1

Pointset Topology

1.1 Basics

At the turn of the twentieth century, Felix Klein suggested that the distinguishing factor between various geometries lay in the kind of transformations allowed before an object is considered “different” from its image under the transformation. (For example, in Euclidean geometry, only rigid transformations result in “congruent” shapes, while in projective geometry, two shapes are “congruent” if they are both views of the same object.) In these notes, we begin the study of *topology*. The grand purpose of topology is to study all “continuous” deformations of objects; therefore, from Klein’s perspective, topology is one of the most basic forms of geometry. This particular course will focus on exactly what is meant by “continuous” functions in settings where few (if any) of the familiar properties of real numbers are valid. Therefore, it is a good idea to start by looking at the “familiar” properties of the real line and sets in general.

In this course, we will take the following as undefined terms:

- POINT
- ELEMENT (or MEMBER)
- SET
- MEMBERSHIP IN A SET

A subset X of a set Y of any set is often called a *pointset* in topology. We will use this term from time to time. As is customary, if we want to indicate that an element x is a member of a set X, we will often write $x \in X$.

For the remainder of this section, we shall be working almost exclusively with subsets of the real numbers. We will denote the set of real numbers by $\mathbb{R}$, and we will think of $\mathbb{R}$ as a line of points stretching forever to the left and to the right of any given point. Important pointsets in $\mathbb{R}$ include the set $\mathbb{Q}$ of rational numbers, the set $\mathbb{Z}$ of integers, and the set $\mathbb{N}$ of natural numbers. There is no universally accepted symbol for the set of irrational numbers.

We will formally recognize one relationship between real numbers: The phrase x *is to the left of* y will mean that x is strictly less than y (in symbols, $x < y$). We will also say that y is *to the right of* x in this situation.

Definition 1.1.1. Let p be a point on the real line and let M be a pointset of $\mathbb{R}$. We say that p is the `leftmost` point of M provided

(1) The point p is a member of M.

(2) If x is a point on the real line to the left of p, then x is not a member of M.

Exercise 1.1.2. Formulate a definition for the rightmost point of a pointset.

Exercise 1.1.3. Formulate the negation of Definition 1.1.1.

In all that follows, we will assume the following properties of the real line. Any other property we use must be a consequence of these axioms.

AXIOMS OF THE REAL LINE

$R1$ If p is a point on the real line, then p is not to the left of p.

$R2$ If p are q are distinct points on the real line, then p is to the left of q or q is to the left of p, but not both.

$R3$ If p and q are points on the real line and p is to the left of q, then there exists a point r on the real line such that p is to the left of r and r is to the left of q.

$R4$ Let p, q, and r be points on the real line. If p is to the left of q and q is to the left of r, then p is to the left of r.

$R5$ If p is a point on the real line, then there exist points q and r on the real line such that q is to the left of p and p is to the left of r.

$R6$ Let U and V be pointsets. The following two conditions together will guarantee that either U has a rightmost point or V has a leftmost point:

1. If x is a point, then $x \in U$ or $x \in V$.
2. If x is a point in U, then x is to the left of every point in V.

The following result, known as the *completeness property*, provides a critical tool when working with real numbers.

Theorem 1.1.4. *Suppose M is a nonempty pointset and b is a point on the real line to the left of every point in M. Either M has a leftmost point, or the set of all points to the left of every point of M has a rightmost point.*

Proof. Let U be the set of all points to the left of every point in M, and let V be the set of all points to the right of every point in U. Note that $M \subseteq V$, and note that U is nonempty, since $b \in U$. If U has a rightmost point, there is nothing to show, so suppose that this is not the case. We want to prove that M has a leftmost point.

To this end, note that U and V satisfy Condition (1) of Axiom $R6$. To see why, suppose that x is a point not in U. It follows that there exist $m \in M$ such that m is to the left of x or $m = x$. Since every point in U is to the left of m by assumption, it follows that x is to the right of every point in U and therefore a member of V. Of course, U and V are chosen to satisfy Condition (2) of Axiom $R6$. Hence, we may conclude that either U has a rightmost point, or that V has a leftmost point. Since we have assumed that U has no rightmost point, it follows that V has a leftmost point, p.

We must now prove that M has a leftmost point. Since $p \in V$, we know that $p \notin U$. If p were to the left of every point in M, then we would have $p \in U$ by construction — a contradiction. Hence, we know that there exist $m \in M$ such that either $p = m$ or m is to the left of p. Now, since p is the leftmost point of V, and since $M \subseteq V$, it follows that there cannot exist $m \in M$ such that m is to the left of p. Therefore, we know that $p = m$. Consequently, we know that $p \in M$ and we know that there is no $m \in M$ to the left of p. We may conclude that p is the leftmost point of M.

□

Definition 1.1.5. Let X be a pointset and let $b \in \mathbb{R}$. We say that b is an *upper bound* for X provided a to the right of b implies $a \notin X$. If the set of upper bounds for X has a leftmost point, we call this point the *least* upper bound for X.

Exercise 1.1.6. Use 1.1.4 to prove the following result: If X is a nonempty pointset with an upper bound, then X has a least upper bound.

Exercise 1.1.7. Suppose U and V are pointsets of $\mathbb{R}$. If U and V both have a leftmost point, prove the union $U \cup V$ also has a leftmost point.

Definition 1.1.8. A pointset S of $\mathbb{R}$ is a `segment` provided there exist points a and b on the real line such that a point x on the real line is a member of S if and only if a is to the left of x and x is to the left of b. We use the symbol (a, b) denote S.

Exercise 1.1.9. Formulate the negation of Definition 1.1.8.

Definition 1.1.10. Let M be a pointset if $\mathbb{R}$ and let p be a point on the real line. We say that p is a `limit point` of M provided every segment containing p also contains a member of M distinct from p.

Exercise 1.1.11. Formulate the negation of Definition 1.1.10.

Exercise 1.1.12. Let a and b be points on the real line such that a is to the left of b, and consider the segment (a, b).

1. Is a a limit point of (a, b)?
2. Is b is limit point of (a, b)?
3. Is x is a member of (a, b), can x be a limit point of this segment?

Exercise 1.1.13. Let M be a pointset of $\mathbb{R}$. If M has a leftmost point p, is p always a limit point of M?

Exercise 1.1.14. Suppose M is a nonempty pointset of $\mathbb{R}$ and b is a point to the left of every point in M. If M has no leftmost point, prove that M has a limit point.

Exercise 1.1.15. Does the set $M = \{1/n : n \text{ is a positive integer}\}$ have a limit point? If so, how many limit points does M possess?

Exercise 1.1.16. Suppose U and V are segments both containing a point x. Use the axioms of the real line to prove $U \cap V$ is a segment.

Exercise 1.1.17. Is the union of two segments necessarily a segment?

Exercise 1.1.18. Suppose N is a pointset of $\mathbb{R}$ and M is a subset of N. If p is a limit point of M, then is p is limit point of N?

Definition 1.1.19. We will say that a pointset M of $\mathbb{R}$ is *open* provided it is the union of a family of segments. We will say that M is *closed* provided $\mathbb{R} - M$ is open.

Exercise 1.1.20. A pointset M of $\mathbb{R}$ is an *interval* provided there exist real numbers a, b such that a is to the left of b and $M = (a, b) \cup \{a, b\}$. We write $M = [a, b]$ in this case. Prove that every interval in $\mathbb{R}$ is closed.

Exercise 1.1.21. Show by example that there exist pointsets of $\mathbb{R}$ which are neither open nor closed.

Exercise 1.1.22. Show by example that there exist pointsets of $\mathbb{R}$ which are both open and closed. (Such sets are called *clopen* sets.)

Exercise 1.1.23. Let $\mathcal{U}$ be a nonempty family of closed pointsets in $\mathbb{R}$.

1. Is the set $\bigcup\{U : U \in \mathcal{U}\}$ necessarily closed?
2. Is the set $\bigcap\{U : U \in \mathcal{U}\}$ necessarily closed?

Exercise 1.1.24. Let $\mathcal{T}$ be a nonempty family of open pointsets of $\mathbb{R}$.

1. Is the set $\bigcup\{T : T \in \mathcal{T}\}$ necessarily open?
2. Is the set $\bigcap\{T : T \in \mathcal{T}\}$ necessarily open?

To conclude this introductory section, we will shift gears for a moment and consider another important concept concerning sets in general.

Definition 1.1.25. Let X and Y be sets.

- We say that X is *less numerous* than Y provided there exists an injection from X to Y but no surjection.
- We say that X is *more numerous* then Y provided there exists a surjection from X to Y but no injection.
- We say that X is *equinumerous* with Y provided there is a bijection between them.

It is common practice to say that equinumerous sets have the same *cardinality*. We say that X is *finite* provided there exists a positive integer n such that X is equinumerous with $Y = \{1, 2, ..., n\}$. We say that X *infinite* if this is not the case.

Theorem 1.1.26. *Every set is less numerous than its powerset.*

Proof. In all that follows, we will let $\mathtt{Su}(X)$ denote the powerset of a set X. There is certainly an injection from X to $\mathtt{Su}(X)$, namely the function $i : X \longrightarrow \mathtt{Su}(X)$ defined by $i(x) = \{x\}$. Suppose by way of contradiction that there exists a mapping $f : X \longrightarrow \mathtt{Su}(X)$ which is onto. Then for every $a \in X$, there is a subset X_a of X such that $f(a) = X_a$. Let

$$Y = \{a \in X : a \notin f(a)\}$$

There must exist $b \in X$ such that $f(b) = Y$. Now, either $b \in Y$ or $b \notin Y$. If $b \in Y$, then $b \notin f(b)$. Since $f(b) = Y$, this is impossible; hence, we must conclude that $b \notin Y$. However, if $b \notin Y$, then $b \notin f(b)$; hence, $b \in Y$ by construction — another impossibility. We must therefore conclude that the function f does not exist.

□

Definition 1.1.27. We say that a set X is *countable* provided it is finite or it is equinumerous with the natural numbers. We say that X is *uncountable* if this is not the case.

An infinite countable set is often called *countably infinite* or *denumerable*. The proof of Theorem 1.1.26 was originally given a century ago by Georg Cantor, now considered the founder of modern set theory; and it certainly makes sense for finite sets. Indeed, it is a routine induction proof to show that the powerset of an n-element set contains exactly 2^n elements. The previous result has deep consequences, however. It proves that uncountable sets must exist; indeed, the powerset of $\mathbb{Z}^+$ must be uncountable. Some years after Cantor presented his proof, Bertrand Russell posed a question which would have a profound effect on Twentieth Century mathematics:

> "What if we consider the *universal set* — that is, the set which contains absolutely everything? How can its powerset be strictly bigger?"

This question, which became known as *Russell's Paradox*, was one of the driving forces behind the construction of formal axioms for set theory in the 1920's and 1930's, as well as the evolution of category theory in the 1950's. It is beyond the scope of these notes to consider either of these endeavors in-depth. We simply note that mathematicians now consider as *sets* only those collections which do not contain themselves as members. Collections for which this cannot necessarily be said are called *proper classes*. Proper classes play an important role in many branches of late Twentieth Century mathematics such as universal algebra, but must be handled with care.

Exercise 1.1.28. Prove that the set of all even positive integers is countably infinite.

Exercise 1.1.29. Prove that the set of all integers is countably infinite.

Theorem 1.1.30. *The union of a countably infinite family of countable sets is countable.*

Proof. Let $\mathcal{F}$ be a countable family of countable sets. Since $\mathcal{F}$ is countably infinite, there is a bijection $f : \mathbb{N} \longrightarrow \mathcal{F}$. For each positive integer n, let $A_n = f(n)$. Then we can write

$$\mathcal{F} = \{A_n : n \in \mathbb{N}\}$$

This is called a *listing* or an *indexing* of the family $\mathcal{F}$. Now, let $A = \bigcup \mathcal{F}$. We want to prove that A is countable; that is, we want to find a bijection $g : \mathbb{N} \longrightarrow A$. Each set A_n is countable; hence, for each positive integer n, there exists a bijection $g_n : \mathbb{N} \longrightarrow A_n$. Now, if $x \in A$, we know $x \in A_n$ for some positive integer n; hence, we know there is a *smallest* positive integer k_x such that $x \in A_{k_x}$. There is a unique positive integer m_x such that $x = g_{k_x}(m_x)$. Define a function $h : A \longrightarrow \mathbb{N}$ by $h(x) = 2^{k_x}3^{m_x}$.

By choosing k_x to be the smallest integer such that $x \in A_x$, we are assured that h is well-defined. We will first prove that h is an injection. To this end, suppose that $h(x) = h(y)$. We need to prove that $x = y$. Now, $h(x) = h(y)$ implies $2^{k_x}3^{m_x} = 2^{k_y}3^{m_y}$. Consequently, we know that $1 = 2^{k_x - k_y}3^{m_x - m_y}$. This tells us that

$$(k_x - k_y)\ln(2) = (m_y - m_x)\ln(3)$$

If $k_x - k_y \neq 0$, then the equation above tells us that

$$\ln\left(\frac{3}{2}\right) = \frac{m_y - m_x}{k_x - k_y}$$

However, this is impossible, since $\ln(3/2)$ is irrational. Consequently, we must have $k_x - k_y = 0$. Similar arguments tell us that $m_y - m_x = 0$. Therefore, we have $k_x = k_y$ and $m_x = m_y$. We may therefore conclude that $x = y$, and this proves that h is an injection.

If we let $h(A)$ denote the range of h, then it is clear that $h^{-1} : h(A) \longrightarrow A$ is a bijection. Construct a function $j : \mathbb{N} \longrightarrow A$ as follows. Let $n \in \mathbb{N}$. If $n \in h(A)$, then let $j(n) = h^{-1}(n)$. If $n \in \mathbb{N} - h(A)$, simply let $j(n) = x$ for any $x \in A$. It follows that j is a surjection; hence, we know that A is countable by Exercise **??**.

□

Exercise 1.1.31. Use Theorem 1.1.30 to prove that the set $\mathbb{Z} \times \mathbb{Z}$ is countable. (For each $n \in \mathbb{N}$, consider the set $A_n = \{n\} \times \mathbb{Z}$.)

As a rule, it is difficult to verify that a set is countable by appealing directly to the definition. Most of the time, we try to find indirect ways to establish whether or not a set is countable. The following result, called the *Cantor-Bernstein Theorem* (or the *Schroder-Bernstein Theorem*) is a fundamental tool in the theory of cardinality and provides such a way. Its proof is rather involved and requires some careful work with functions.

Theorem 1.1.32. *Suppose X and Y are sets. If X is equinumerous with a subset of Y and Y is equinumerous with a subset of X, then X and Y have the same cardinality.*

Proof. To prove that X nd Y are equinumerous, we must prove there is a bijection between them. Since X is equinumerous with a subset of Y, we know there must exist a function $f : X \longrightarrow Y$ which is one-to-one. Likewise, there exists a function $g : Y \longrightarrow X$ which is one-to-one. Let $g(Y)$ denote the ranges of g and suppose that $x \in X$. Call $g^{-1}(x)$ and $f^{-1}(y)$ the *first ancestor* of x (if it exists) under the function g. Proceeding in like fashion, call $f^{-1}(g^{-1}(x))$ the *second ancestor* of x, $g^{-1}(f^{-1}(g^{-1}(x)))$ the *third ancestor* of x, and so on. There are three possibilities:

1. x could have infinitely many ancestors
2. x could have an even (possibly zero) number of ancestors
3. x could have an odd number of ancestors

With this in mind, let

$$X_i = \{x \in X : x \text{ has infinitely many ancestors}\}$$
$$X_e = \{x \in X : x \text{ has an even number of ancestors}\}$$
$$X_o = \{x \in X : x \text{ has an odd number of ancestors}\}$$

Notice that the set X_e contains $X - g(Y)$. In like fashion, construct the sets Y_i, Y_e, and Y_o from the ancestors of elements in Y under the function f. Consider the new function $h : X \longrightarrow Y$ defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in X_i \cup X_e; \\ g^{-1}(x), & \text{if } x \in X_o. \end{cases}$$

Our goal will be to prove that h is a bijection. Notice that X_i, X_e, and X_o must be pairwise disjoint; hence, h is a well-defined function. Since both f and g^{-1} are one-to-one; we know that h is one-to-one. Consequently, all we really must do is prove that h is onto. To this end, suppose that $y \in Y$. We must prove that there exist $x \in X$ such that $h(x) = y$. We know that y must be a member of Y_i, or a member of Y_e, or a member of Y_o.

Suppose that $y \in Y_i$. This means that y has infinitely many ancestors under f. Consider $x = f^{-1}(y)$. We will prove that $x \in X_i$ (which would imply that $h(x) = f(f^{-1}(y)) = y$). Since y has infinitely many ancestors, it must be true that $g^{-1}(x) = g^{-1}(f^{-1}(y))$ exists. In particular, the second ancestor of y must be the *first* ancestor of x. Generalizing, if $n > 1$, then the nth ancestor of y must be the $n-1$ ancestor of x; and we may conclude that $x \in X_i$, as desired.

Suppose that $y \in Y_o$. Consider $x = f^{-1}(y)$. We will prove that $x \in X_e$ (which would imply that $h(x) = f(f^{-1}(y)) = y$). There exists an odd integer n such that y has exactly n ancestors. It follows that the nth ancestor of y is also the $n-1$st ancestor of x. Since the nth ancestor of x would be the $n+1$st ancestor or y, we must conclude that the nth ancestor of x does not exist. Hence, we must conclude that $x \in X_e$, as desired.

Finally, suppose that $y \in Y_e$ and consider $x = g(y)$. We will prove that $x \in X_o$ (which would imply that $h(x) = g^{-1}(g(y)) = y$). Either y has an ancestor under f, or it does not. If it does not, then $x = g(y)$ has exactly one ancestor under g, namely y; and we may conclude that $x \in X_o$, as desired.

Suppose y has an ancestor under f. It follows that there is an even integer $n > 0$ such that y has exactly n ancestors. Since y is the first ancestor of x (because $y = g^{-1}(x)$), we may conclude that the *first* ancestor of y must be the *second* ancestor of x. Consequently, we know the nth ancestor of y must be the $n+1$st ancestor of x. The $n+2$nd ancestor of x cannot exist (since it would be the $n+1$st ancestor of y); hence, we must conclude that $x \in X_o$, as desired.

□

Exercise 1.1.33. Use the Cantor-Bernstein Theorem to prove that every infinite subset of a countable set is countable.

Exercise 1.1.34. Let X and Y be sets and suppose $f : X \longrightarrow Y$ is a surjection. If X is countable, prove that Y is also countable.

Exercise 1.1.35. Use Exercises 1.1.31 and 1.1.33 to prove that the set of rational numbers is countable.

Theorem 1.1.36. *The set of real numbers is uncountable.*

Proof. We will prove this result by contradiction. Suppose that $\mathbb{R}$ is countable. We can index the set $\mathbb{R}$ by the natural numbers (see the proof of Theorem 1.1.30). In particular, let $\mathbb{R} = \{x_1, ..., x_n, ...\}$. (At this point, we are ignoring the natural ordering on $\mathbb{R}$.)

Consider the element $c_1 = x_2$. Clearly there exist real numbers to the right of c_1. Since $\mathbb{N}$ is well-ordered, there exists a smallest natural number k_1 such that $c_1 < x_{k_1}$. Let $d_1 = x_{k_1}$. Now, there must exist real numbers α such that $c_1 < \alpha < d_1$. Let c_2 be the number α with smallest index.

If we consider c_2 and d_1, we can repeat the process above to select new real numbers c_3 and d_2 such that

$$c_1 < c_2 < c_3 < d_2 < d_1$$

In fact, we can keep repeating this process indefinitely, creating two sequences $A = \{c_j : j \in \mathbb{N}\}$ and $B = \{d_j : j \in \mathbb{N}\}$ with the following properties:

- We have $c_j < c_k$ and $d_j < d_k$ for all $j < k$.
- We have $c_j < d_k$ for all natural numbers j and k.

Now, by construction, c_1 is to the left of every member of the pointset B; hence, we know by Theorem 1.1.4 that B either has a leftmost point, or the set of all points to the left of B has a rightmost point. By construction, we know that B does not have a leftmost point; hence, we must conclude that the set S of all points to the left of B has a rightmost point. Call this point c. Clearly, we must have c to the right of every point in the sequence A. Furthermore, since A does not have a rightmost point by construction, we know that $c \notin A$.

Since $\mathbb{R}$ is assumed countable, we must have $c = x_m$ for some positive integer m. For each positive integer n, there must exist $d_p = x_{k_p} \in B$ such that $n < k_p$. Since $c \notin A$, we know that $c \neq c_{p+1} = x_{j_{p+1}}$. But, by construction, we know that c_{p+1} is the *first* element in the indexing of $\mathbb{R}$ with the property that $c_{p+1} < d_p$. Therefore, we know that $k_p < j_{p+1} < m$. Consequently, we must have m strictly larger than every positive integer — an impossibility.

□

Exercise 1.1.37. Let $H \subseteq \mathbb{R}$. If for each $x \in H$ there exist $\epsilon_x > 0$ such that $(x, x+\epsilon_x) \cap H = \emptyset$, then prove that H is countable. Hint: Let $p_x \in (x, x+\epsilon_x) \cap \mathbb{Q}$ and consider the function $f : H \longrightarrow \mathbb{Q}$ defined by $f(x) = p_x$. Show that f is an injection.

1.2 Introduction to Topology

In analysis, you saw that continuity for functions on the real line is intimately tied up with the notion of a segment (the $\epsilon - \delta$ definition from calculus). This will be important if we are to carry the idea of continuity into more general settings. Segments on the real line are "basic" open sets in the sense that every other open set can be built up as unions of segments; hence, our study of continuity begins with generalizations of segments and the sets built up from them.

Definition 1.2.1. Let X be any set and let Ω be a nonempty family of subsets from X. We say that Ω is a *topology* on X provided the following conditions hold:

1. The empty set is a member of Ω;
2. The set X is a member of Ω;
3. If $A_1, A_2 \in \Omega$, then $A_1 \cap A_2 \in \Omega$;
4. If $\mathcal{F}$ is any nonempty family of Ω, then $\bigcup\{F : F \in \mathcal{F}\}$ is a member of Ω.

Given a set X and a topology Ω on X, we call the pair (X, Ω) a *topological space* and refer to the members of Ω as *opens*. Notice that the properties of a topology on a set are simply the properties we encountered for open pointsets of the real line.

Exercise 1.2.2. Let X be any set. Show that the following are topologies for X:

1. The *discrete topology* $\Omega = \mathtt{Su}(X)$ (In other words, Ω is the powerset for X.)

2. The *indiscrete topology* $\Omega = \{X, \emptyset\}$

Definition 1.2.3. Let X be any set and let Ω and Θ be topologies on X. We say that Θ is *finer* than Ω provided $\Omega \subset \Theta$. We also say that Ω is *coarser* than Θ if this is the case.

Often times we try to find the coarsest topology on X that allows us to perform whatever operation we are interested in.

Exercise 1.2.4. Let $\mathbb{R}$ be the set of real numbers. Declare $U \subseteq \mathbb{R}$ to be open provided one of the following conditions holds:

1. $U = \emptyset$
2. For all $x \in U$, there exists a segment $I = (a, b)$ such that $x \in I$ and $I \subseteq U$.

Show that this is a topology on $\mathbb{R}$. (This is called the *usual* topology on $\mathbb{R}$, and we will denote it by Ω_U.)

Exercise 1.2.5. For the real numbers $\mathbb{R}$, show that the following family is a topology:

$$\Omega_M = \{U \cup F : U \in \Omega_U \text{ and } F \subseteq \mathbb{R} - \mathbb{Q}\}$$

The space $(\mathbb{R}, \Omega_M)$ is sometimes called the *Michael Line*.

Exercise 1.2.6. Let X be any set. A subset A of X is *cofinite* provided $X - A$ is finite. Show that the set $\texttt{Cof}(X)$ of all cofinite subsets of X (along with the empty set) forms a topology. (This is called the *cofinite* topology on X.)

Exercise 1.2.7. For the real numbers $\mathbb{R}$, show that the following family is a topology:

$$\Omega_F = \{A \in \texttt{Su}(\mathbb{R}) : 0 \notin A\} \cup \{A \in \texttt{Su}(\mathbb{R}) : 0 \in A \text{ and } \mathbb{R} - A \text{ is finite}\}$$

Exercise 1.2.8. For the real numbers $\mathbb{R}$, show that the following family is a topology:

$$\Omega_C = \{A \in \texttt{Su}(\mathbb{R}) : 0 \notin A\} \cup \{A \in \texttt{Su}(\mathbb{R}) : 0 \in A \text{ and } \mathbb{R} - A \text{ is countable}\}$$

Definition 1.2.9. Let (X, Ω) be a topological space. A subset C of X is *closed* relative to Ω provided $X - C \in \Omega$. We will let $\kappa(\Omega)$ denote the set of all subsets of X closed relative to Ω.

Exercise 1.2.10. Let (X, Ω) be a topological space. Use DeMorgan's Laws to prove that the family $\kappa(\Omega)$ satisfies the following conditions:

1. We have $\emptyset \in \kappa(\Omega)$.
2. We have $X \in \kappa(\Omega)$.
3. If $C_1, ..., C_n \in \kappa(\Omega)$, then $C_1 \cup ... \cup C_n \in \kappa(\Omega)$.
4. If $\mathcal{C} \subseteq \kappa(\Omega)$ is nonempty, then $\bigcap\{C : C \in \mathcal{C}\} \in \kappa(\Omega)$.

Definition 1.2.11. Let X be any set and let Ω be a topology on X. A *basis* for Ω is a family $\mathcal{B}_\Omega \subseteq \Omega$ such that every member of Ω is the union of a subcollection from $\mathcal{B}$.

We say that a basis for a topology *generates* that topology. Every topology is a basis for itself. We call the members of a given basis *basic opens* for X under Ω.

Let X be any set, and let Ω and Θ be topologies on X. We say that the topological spaces (X, Ω) and (X, Θ) are *equal* provided $\Omega = \Theta$. Two bases for topologies on X are *equivalent* if they generate equal topological spaces.

Theorem 1.2.12. *Let X be any set, and suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for topologies on X. These bases are equivalent if and only if,*

1. *Whenever $B \in \mathcal{B}_1$ and $p \in B$, there exist $A \in \mathcal{B}_2$ such that $p \in A \subseteq B$; and*
2. *Whenever $A \in \mathcal{B}_2$ and $q \in A$, there exist $B \in \mathcal{B}_1$ such that $q \in B \subseteq A$.*

Proof. Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent. It follows that $\Omega_1 = \Omega_2$. Let $K \in \mathcal{B}_1$, and let $p \in K$. Since $\Omega_1 = \Omega_2$, and since $K \in \Omega_1$ by assumption, it follows that $K \in \Omega_2$. Thus, we know there exists a family $\mathcal{F} \subseteq \mathcal{B}_2$ such that $K = \bigcup \mathcal{F}$. Consequently, there exist $A \in \mathcal{F}$ such that $p \in A$. Clearly we have $A \subseteq K$ as well. Thus, Condition (1) holds. The proof that Condition (2) holds is, of course, similar.

On the other hand, suppose that Conditions (1) and (2) hold. We must prove that $\Omega_1 = \Omega_2$. Let $A \in \Omega_1$. There exists a family $\mathcal{U} \subseteq \mathcal{B}_1$ such that $A = \bigcup \mathcal{U}$. Let $p \in A$. There exist $U_i \in \mathcal{U}$ such that $p \in U_i$. By Condition (1), there exist $V_i \in \mathcal{B}_2$ such that $p \in V_i \subseteq U_i$. Let $\mathcal{V} = \{V_p : p \in A\}$ and consider $C = \bigcup \mathcal{V}$. Clearly we have $A \subseteq \bigcup \mathcal{V}$. However, since each member of $\mathcal{V}$ is a subset of A, it also is clear that $\bigcup \mathcal{V} \subseteq A$.

It follows that $A \in \Omega_2$; and we may conclude that $\Omega_1 \subseteq \Omega_2$. The proof that $\Omega_2 \subseteq \Omega_1$ is similar.

□

Exercise 1.2.13. Show that the usual topology for $\mathbb{R}$ has a countable basis.

Theorem 1.2.14. *Let X be any set and let $\mathcal{B} \subseteq \mathtt{Su}(X)$. The family $\mathcal{B}$ is a basis for a topology on X if and only if the following conditions are met.*

1. *We have $X = \bigcup \mathcal{B}$.*

2. *If $U, V \in \mathcal{B}$ and $x \in U \cap V$, then there exist $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$.*

Proof. Suppose first that $\mathcal{B}$ is a basis for some topology Ω on X. It follows that $X \in \Omega$; hence, the definition of basis tells us that X is the union of some subcollection of $\mathcal{B}$. Of course, this implies that $X = \bigcup \mathcal{B}$. Now, suppose that $x \in U \cap V$ for some $U, V \in \mathcal{B}$. By assumption, U and V are open; hence, $U \cap V$ is open. Since every open set is the union of some subcollection from $\mathcal{B}$, we know there exists a family $\mathcal{F} \in \mathtt{Su}(\mathcal{B})$ such that $U \cap V = \bigcup \mathcal{F}$. Of course, this means there must exist some $W \in \mathcal{F}$ such that $x \in W \subseteq U \cap V$.

Conversely, suppose that Conditions (1) and (2) hold. We must identify a topology Ω that is generated by $\mathcal{B}$. The simplest choice to consider is the family $\Omega = \{\bigcup \mathcal{F} : \mathcal{F} \in \mathtt{Su}(\mathcal{B})\}$. Note that $X \in \Omega$ by Condition (1). Since $\emptyset \in \mathtt{Su}(\mathcal{B})$, we know that $\emptyset = \bigcup \emptyset \in \Omega$. The family Ω is certainly closed under arbitrary unions. We need only prove that Ω is closed under finite intersections. To this end, let A_1, A_2 be members of Ω and let $A = A_1 \cap A_2$. We must prove that $A \in \Omega$.

By assumption, there exist $\mathcal{F}_1, \mathcal{F}_2 \in \mathtt{Su}(\mathcal{B})$ such that $A_1 = \bigcup \mathcal{F}_1$ and $A_2 = \bigcup \mathcal{F}_2$. Furthermore, we know from basic set theory that

$$A = \bigcup \mathcal{F}_1 \cap \bigcup \mathcal{F}_2 = \bigcup \{B \cap C : B \in \mathcal{F}_1, C \in \mathcal{F}_2\}$$

Consequently, for each $x \in A$, there must exist $B_x \in \mathcal{F}_1$ and $C_x \in \mathcal{F}_2$ such that $x \in B_x \cap C_x$. Now, by Condition (2), we may assume there exist $W_x \in \mathcal{B}$ such that $x \in W_x \subseteq B_x \cap C_x$. If we let

$$\mathcal{F} = \{W_x : x \in A\}$$

then it is clear that $\mathcal{F} \in \mathtt{Su}(\mathcal{B})$ and $A = \bigcup \mathcal{F}$. Hence, we may conclude that $A \in \Omega$, as desired.

□

Exercise 1.2.15. Let $\mathbb{R}$ be the set of real numbers. Show that the set $\mathcal{B}_L = \{[a, b) : a < b \in \mathbb{R}\}$ is a basis for a topology on $\mathbb{R}$. (This topology is called the *Sorgenfrey Line* and will be denoted by Ω_S.)

Exercise 1.2.16. Consider the Michael Line $(\mathbb{R}, \Omega_M)$ defined in Exercise 1.2.5 and let

$$\mathcal{B}_M = \{(a, b) : a, b \in \mathbb{R} - \mathbb{Q} \text{ and } a < b\} \cup \{\{c\} : c \in \mathbb{R} - \mathbb{Q}\}$$

1. Prove that $\mathcal{B}_M$ is a basis for the Michael Line.

2. Prove that every member of $\mathcal{B}_M$ is both open *and* closed in the Michael Line. (Such sets are called *clopen*.)

Definition 1.2.17. Let (X, Ω) be any topological space and let $x \in X$. A subset N of X is called a *neighborhood* of x relative to Ω provided there exist $U \in \Omega$ such that $x \in U \subseteq N$.

Exercise 1.2.18. Let (X, Ω) be any topological space. And let $U \subseteq X$. Show that $U \in \Omega$ if and only if it is a neighborhood to each of its members.

Definition 1.2.19. Let (X, Ω) be any topological space and let $A \subseteq X$. We say a point $x \in X$ is a *limit point* of A relative to Ω provided every neighborhood of x contains a point of A other than x. We will let $\overline{A}$ denote the union of A with its set of limit points and call this set the *closure* of A with respect to Ω.

Exercise 1.2.20. Let (X, Ω) be any topological space and let A, B be subsets of X such that $A \subseteq B$. Show that every limit point of A is a limit point of B. Is the converse necessarily true?

Theorem 1.2.21. *Let (X, Ω) be any topological space. A subset A of X is closed relative to Ω if and only if $A = \overline{A}$.*

Proof. If A is closed, then $X - A$ is open; this means that $X - A$ is a neighborhood to each of its members by Exercise 87. Consequently, no member of $X - A$ can be a limit point for A, and A must contain all of its limit points. It now follows that $A = \overline{A}$.

Conversely, suppose that $A = \overline{A}$. It follows that A contains all of its limit points. Let $p \in X - A$. Since p is not a limit point of A, we know that there exists a neighborhood N_p containing p that contains no member of A. Consequently, $N_p \subseteq X - A$; and it follows that $X - A$ is a neighborhood to each of its members. Thus, $X - A$ is open; and we may conclude that A is closed.

□

Theorem 1.2.22. *Let (X, Ω) be any topological space. If $A \subseteq X$, then*

$$\overline{A} = \bigcap \{C \in \kappa(\Omega) : A \subseteq C\}$$

Exercise 1.2.23. Prove Theorem 1.2.22.

Theorem 1.2.24. *If (X, Ω) is any topological space and A, B are subsets of X, then the following are true:*

1. *We have $\overline{A} = \overline{(\overline{A})}$.*

2. *We have* $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. By Theorem 1.2.22, we know that $\overline{A}$ is closed. Consequently, $\overline{A}$ must contain all of its limit points; and this implies that $\overline{(\overline{A})} = \overline{A}$.

Suppose now that A and B are subsets of X. We know that every limit point of A is a limit point of $A \cup B$, and every limit point of B is a limit point of $A \cup B$ by Exercise 1.2.20. Thus, we know that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Now, since $\overline{A}$ and $\overline{B}$ are closed, we know that $\overline{A} \cup \overline{B}$ is also closed. Since $A \cup B \subseteq \overline{A} \cup \overline{B}$, Theorem **??** tells us that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

□

Definition 1.2.25. Let (X, Ω) be any topological space and let $A \subseteq X$. The *interior* of A relative to Ω is the set

$$A^\circ = \{x \in A : x \in U \subseteq A \text{ for some } U \in \Omega\}$$

Theorem 1.2.26. *If* (X, Ω) *is any topological space and* $A, B \subseteq X$*, then the following are true:*

1. *We have* $\overline{X - A} = X - A^\circ$.
2. *We have* $(A^\circ)^\circ = A^\circ$.
3. *We have* $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Exercise 1.2.27. Prove Theorem 1.2.26.

Definition 1.2.28. Let (X, Ω) be any topological space and let $A \subseteq X$. The *boundary* of A relative to Ω is the set

$$\mathtt{Bdy}(A) = \{x \in X : x \notin A^\circ \text{ and } x \notin (X - A)^\circ\}$$

Exercise 1.2.29. Let (X, Ω) be any topological space and let $A \subseteq X$. Show that the following holds relative to Ω.

$$\mathtt{Bdy}(A) = \{x \in X : N \cap A \neq \emptyset \text{ and } N \cap (X - A) \neq \emptyset \text{ for all neighborhoods } N \text{ of } x\}$$

Exercise 1.2.30. Let (X, Ω) be any topological space and let $A \subseteq X$. Show that the following statements hold relative to Ω.

1. The set A is closed if and only if $\mathtt{Bdy}(A) \subseteq A$.
2. The set A is open if and only if $A \cap \mathtt{Bdy}(A) = \emptyset$.

Exercise 1.2.31. Let (X, Ω) be any topological space and let $A \subseteq X$. Show that the following statements hold relative to Ω.

1. We have $A^\circ = A - \mathtt{Bdy}(A)$.
2. We have $\overline{A} = A \cup \mathtt{Bdy}(A)$.
3. We have $X - \mathtt{Bdy}(A) = A^\circ \cup (X - A)^\circ$.
4. We have $\mathtt{Bdy}(A) = \overline{A} \cap \overline{X - A} = \overline{A} - A^\circ$.

We will conclude this section with a simple way to construct new topological spaces from existing ones. We will consider other methods in a later section.

Exercise 1.2.32. Let (X, Ω) be a topological space, and let $Y \subseteq X$. Prove that the collection $\Omega_Y = \{Y \cap U : U \in \Omega\}$ forms a topology on Y.

The family Ω_Y appearing in the last exercise is called the *subspace topology* on the set Y.

Exercise 1.2.33. Let (X, Ω) be a topological space, and let (Y, Ω_Y) be a subspace of (X, Ω).

1. If B is a basis for Ω, prove that $B_Y = \{Y \cap V : V \in B\}$ is a basis for Ω_Y.
2. If $Y \in \Omega$, prove that $\Omega_Y \subseteq \Omega$.
3. If A is closed in Y and Y is closed in X, prove that A is closed in X.

Exercise 1.2.34. Consider $\mathbb{R}$ under the usual topology. Characterize the open sets of the interval $[-1, 1]$ under the subspace topology.

Exercise 1.2.35. Let (X, Ω) be a topological space and let (Y, Ω_Y) be a subspace of (X, Ω). If $A \subseteq Y$, prove that the closure of A in Y is the set $\overline{A} \cap Y$.

Exercise 1.2.36. Let (X, Ω) be a topological space and suppose that $\mathcal{T} = \{(T_i, \Omega_i) : i \in I\}$ is a family of subspaces of (X, Ω). Prove that the subspace topology on $\bigcup\{T_i : i \in I\}$ is the family

$$\Omega' = \{\bigcup\{T_i \cap U : i \in I\} : U \in \Omega\}$$

1.3 Continuity

In this section, we will introduce the notion of continuity, one of the key concepts in topology. We will begin by recalling the classic definition of continuity given in calculus.

Definition 1.3.1. Let $M \subseteq \mathbb{R}$ be a nonempty and open in the usual topology, let $f : M \longrightarrow \mathbb{R}$ be a function, and let $a \in M$. We say that f is *continuous* at a provided for every $\epsilon > 0$, there exist $\delta > 0$ such that $|x - a| < \delta$ guarantees that $|f(x) - f(a)| < \epsilon$. We say that f is *continuous on* M provided f is continuous at every point in M.

Here, we are requiring that M be open simply to avoid having to discuss left and right continuity as part of the definition. The first question we will consider is how this definition can be recast using on the language of the usual topology on $\mathbb{R}$. This will give us a way to extend the ideal of continuity to any topological space.

First, notice that the absolute value inequalities appearing in the classic definition actually give us segments (basic open sets in the usual topology). In particular, the solution set to the inequality $|x - a| < \delta$ is really the segment $(a - \delta, a + \delta)$. Likewise, the solution set for $|f(x) - f(a)| < \epsilon$ is the segment $(f(a)-\epsilon, f(a)+\epsilon)$. In essence, then, the classic definition is saying that, whenever we have a segment in $V \subseteq \mathbb{R}$ containing $f(a)$, we can find a segment $U \subseteq M$ such that

$$f(U) = \{f(x) : x \in U\} \subseteq V$$

With this in mind, consider the following definition.

Definition 1.3.2. Let M be a pointset and suppose that $f : M \longrightarrow \mathbb{R}$ is a function. We say that f is `continuous` at a point $p \in M$ *relative to the usual topology* provided for every segment V containing $f(p)$, there exists a segment U containing p such that $f(U \cap M) \subseteq V$. Furthermore, we say that f is *continuous* on M provided f is continuous at all $p \in M$.

Notice that we are not requiring M to be open in the usual topology, and notice what we did instead — we require that there exist a segment U such that *every point in* U *that is also in* M is mapped to V by f.

Exercise 1.3.3. Use Definition 1.3.2 to prove the following. If M is a pointset having no limit point, then every function $f : M \longrightarrow \mathbb{R}$ is continuous on M under the usual topology.

Definition 1.3.4. Let M be a pointset and let $f : M \longrightarrow \mathbb{R}$ be a function. Let $\lambda \in \mathbb{R}$ and let $p \in M$. We say that f has *limit* λ *at the point* p provided for every segment T containing λ, there exists a segment S containing p such that $f(S \cap M)$ is a subset of T.

Exercise 1.3.5. Let M be a pointset and let $p \in M$. Prove that a function $f : M \longrightarrow \mathbb{R}$ is continuous at p relative to the usual topology if and only if f has limit $f(p)$ at p.

The previous exercise is the usual "limit definition" of continuity seen in Calculus I. This exercise shows that Definition 1.3.2 is logically equivalent to the one used in elementary calculus.

Exercise 1.3.6. Let M be a pointset and suppose that $f : M \longrightarrow \mathbb{R}$ is a function. Prove that f is continuous on M relative to the usual topology if and only if $f^{-1}(T) = \{x \in U : f(x) \in T\}$ is open relative to the subspace topology whenever T is open.

Exercise 1.3.7. Let M be a pointset and suppose that $f : M \longrightarrow \mathbb{R}$ is a continuous function relative to the usual topology. If $N \subseteq M$, show that the function

$$f_N = \{(n, f(n)) : n \in N\}$$

is continuous on N. We call f_N the *restriction* of f to N.

Exercise 1.3.8. Let M be a pointset and let $f : M \longrightarrow \mathbb{R}$ be a function. Let $N \subseteq f(M)$ and let $g : N \longrightarrow \mathbb{R}$ be a function. Let

$$g \circ f = \{(m, g(f(m))) : m \in f^{-1}(N)\}$$

1. Show that $g \circ f$ is a function from $f^{-1}(N)$ to $\mathbb{R}$.

2. If f and g are continuous on their respective domains, show that $g \circ f$ is continuous on $f^{-1}(N)$ relative to the usual topology.

Let x and y be distinct points on the real line. We will say that a point w is *between* x and y provided

- $x < w < y$; or

- $y < w < x$.

The following result, sometimes called the *Intermediate Value Theorem*, is one of the most important results in elementary calculus. It is generally used without proof in that context, but we now have the ability to establish it.

Theorem 1.3.9. *Let $[a, b]$ be an interval and suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function relative to the usual topology. Suppose that x and y are members of $[a, b]$. If $f(x) \neq f(y)$, then for every point w between $f(x)$ and $f(y)$, there is a point z between x and y such that $f(z) = w$.*

Proof. Since f is a function, we know that x is distinct from y; consequently, there exist points z between x and y. Suppose by way of contradiction that there exists a point w between $f(x)$ and $f(y)$ such that $f^{-1}(\{w\}) \cap [x, y] = \emptyset$. We will contradict the assumption that f is continuous on $[a, b]$.

Since x and y are in $[a, b]$, we know that every point between x and y is also a member of $[a, b]$. By assumption, we therefore know $f(z)$ exists, and we know that f is continuous at z for all z between x and y. For the sake of simplicity, assume that x is left of y and $f(x)$ is left of $f(y)$. Arguments for the other cases are similar. Let M denote the set of all points q in $[x, y]$ such that $f(q)$ is to the left of w, and let N denote the set of all points to the right of every point in M. Since $f(x)$ is to the left of w, we know M is nonempty. Furthermore, we know that y is a point to the right of every point in M since q is assumed to be to the left of y. Consequently, we know by Theorem 1.1.4 that M is a rightmost point, or N has a leftmost point. We consider each possibility separately.

Suppose first that M has a rightmost point, ρ. By Theorem 1.1.32, we know that ρ is a limit point for N. Since $\rho \in M$, we know that $f(\rho)$ is to the left of w. Let c be a point to the left of $f(\rho)$ and consider the segment (c, w). Since ρ is a limit point for N, we know every segment S containing ρ must contain a member of N. Call this point v_S. Since $v_S \in N$, we know that $v_S \notin M$; hence, we know that $f(v_S)$ is not to the left of w. This implies that $f(v_S) \notin (c, w)$. Consequently, we know that every segment S containing ρ is such that $f(S) \not\subseteq (c, w)$ — contradicting the assumption that f is continuous at ρ.

Suppose instead that the set N has a leftmost point, λ. By Theorem 1.1.32, we know that λ is a limit point for M. If λ were to the right of y, then any segment (y, c) containing λ would also contain a member of M — contrary to the construction of M. Hence, we know that $\lambda \in [x, y]$. By our assumption that $w \neq f(z)$ for any $z \in [a, b]$, it follows that $f(\lambda) \neq w$. Since $\lambda \notin M$, we know therefore know that w is to the left of $f(\lambda)$. Let c be a point to the right of $f(\lambda)$ and consider the segment (w, c). Since λ is a limit point of M, every segment S containing λ must contain a member of M. Call this point v_S. Since $v_S \in M$, we know that $f(v_S)$ lies to the left of w and therefore is nota member of (w, c). Therefore, every segment S containing λ has the property that $f(S) \not\subseteq (w, c)$ — contradicting our assumption that f is continuous at λ.

We have no choice now but to conclude that it is impossible to construct the set M. Consequently, there does not exist a point w between $f(x)$ and $f(y)$ such that $f^{-1}(\{w\}) \cap [x, y] = \emptyset$.

□

At this point, we are ready to introduce the general topology definition of continuity. Notice that it is based on Exercise 1.3.6.

Definition 1.3.10. Let (X, Ω) and (Y, Θ) be topological spaces. A function $f : X \longrightarrow Y$ is *continuous* relative to Ω and Θ provided $f^{-1}(U) \in \Omega$ for every $U \in \Theta$.

Theorem 1.3.11. *Let (X, Ω) and (Y, Θ) be topological spaces and let $f : X \longrightarrow Y$ be a function. The following statements are equivalent:*

1. *The function f is continuous relative to Ω and Θ.*
2. *We have $f^{-1}(C) \in \kappa(\Omega)$ for every $C \in \kappa(\Theta)$.*
3. *The set $f^{-1}(N)$ is a neighborhood of x for every $x \in X$ and neighborhood N of $f(x)$.*
4. *For each $x \in X$ and neighborhood N of $f(x)$, there exists a neighborhood M of x such that $f(M) \subseteq N$.*
5. *For each $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.*
6. *For each $B \subseteq Y$, we have $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.*

Proof. We first prove that Claim (1) implies Claim (2). To this end, suppose that f is continuous and let $C \in \kappa(\Theta)$. It follows that $X - C \in \Theta$; hence, we know that $f^{-1}(Y - C) \in \Omega$. We need only prove that $f^{-1}(Y - C) = X - f^{-1}(C)$. To this end, suppose that $a \in f{-}1(Y - C)$. It follows that $f(a) \in Y - C$, which, of course, means that $f(a) \notin C$. Consequently, we know that $a \notin f^{-1}(C)$. We may therefore conclude that $f^{-1}(Y - C) \subseteq X - f^{-1}(C)$. On the other hand, suppose that $b \in X - f^{-1}(C)$. It follows that $b \notin f^{-1}(C)$, which implies that $f(b) \notin C$. This tells us that $f(b) \in Y - C$; hence, we know that $b \in f^{-1}(Y - C)$. Thus, we may conclude that $X - f{-}1(C) \subseteq f^{-1}(Y - C)$, as desired.

We now prove that Claim (2) implies Claim (3). To this end, suppose that $f{-}1(C) \in \kappa(\Omega)$ for each $C \in \kappa(\Theta)$. Let $x \in X$ and let N be a neighborhood of $f(x)$ in Y. It follows that there exists $U \in \Theta$ such that $f(x) \in U \subseteq N$. Consider the set $Y - U$. By assumption, we know that $X - f^{-1}(U) = f^{-1}(Y - U) \in \kappa(\Omega)$. Consequently, we know that $f^{-1}(U) \in \Omega$. Since $x \in f^{-1}(U) \subseteq f^{-1}(N)$, we are done.

Observe that Claim (4) is an immediate consequence of Claim (3) (just let $M = f^{-1}(U)$, where U is the open in the neighborhood N containing $f(x)$ such that $f(x) \in U \subseteq N$ for example).

We will prove that Claim (4) implies Claim (5). To this end, let $A \subseteq X$. Consider the sets $f(\overline{A})$ and $\overline{f(A)}$. Note that $f(\overline{A}) \subseteq \overline{f(A)}$ if and only if $Y - \overline{f(A)} \subseteq Y - f(\overline{A})$. Let $b \in Y - \overline{f(A)}$. If $f^{-1}(b) = \emptyset$, then it is clear that $b \notin f(\overline{A})$. Suppose instead that $f^{-1}(b)$ is nonempty and let $x \in f^{-1}(b)$. Since

$Y - \overline{f(A)}$ is a neighborhood for b, there exists a neighborhood M of x such that $f(M) \subseteq Y - \overline{f(A)}$. It follows that $M \cap A = \emptyset$; thus, $x \notin A$ and x is not a limit point for A. Therefore, we know that$x \notin \overline{A}$, which implies that $b \notin f(\overline{A})$. Thus, whether or not $f^{-1}(b) = \emptyset$, we know that $b \in Y - f(\overline{A})$, as desired.

We now prove that Claim (5) implies Claim (6). To this end, let $B \subseteq Y$ and consider the sets $f^{-1}(\overline{B})$ and $\overline{f^{-1}(B)}$. By Claim (5), we know that

$$f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} = \overline{B}$$

Consequently, we know that

$$\overline{f^{-1}(B)} = f^{-1}(f(\overline{f^{-1}(B)})) \subseteq f^{-1}(\overline{B})$$

as desired.

Finally, we prove that Claim (6) implies Claim (1). To this end, suppose that $y \in Y$ and suppose $U \in \Theta$ contains y. We must prove that $f^{-1}(U) \in \Omega$. We know by previous arguments and Claim (6) that

$$\overline{X - f^{-1}(U)} = \overline{f^{-1}(Y - U)} \subseteq f^{-1}(\overline{Y - U}) = f^{-1}(Y - U) = X - f^{-1}(U)$$

Thus, since $X - f^{-1}(U)$ is clearly contained in $\overline{X - f^{-1}(U)}$, it follows that $X - f^{-1}(U)$ must be closed. Thus, $f^{-1}(U)$ is open.

□

1.4 Separation Axioms

In this section, we introduce ways to measure "how many" open sets a particular topology contains. The so-called *separation axioms* provide a way to tell when a given topology has enough open sets to distinguish between individual points, to distinguish between points and closed sets, and to distinguish between closed sets.

Definition 1.4.1. `(Separation Axioms)` Let (X, Ω) be any topological space.

1. We say that (X, Ω) is T_0 if, for each distinct $x, y \in X$, there exists a neighborhood M of x such that $y \notin M$ or there exists a neighborhood M of y such that $x \notin M$.

2. We say that (X, Ω) is T_1 if, for each distinct $x, y \in X$, there exist neighborhoods M and N of x and y, respectively, such that $y \notin M$ and $x \notin N$.

3. We way that (X, Ω) is T_2 if, for each distinct $x, y \in X$, there exist disjoint neighborhoods of x and y.

The T's used in enumerating the separation axioms come from the German word *trennungsaxiome*, which simply means "separation axiom." Spaces exhibiting the T_2 separation axiom are often called *Hausdorff* spaces.

Exercise 1.4.2. Let $X = \{0, 1\}$ and let $\Omega = \{\emptyset, \{0\}, \{0, 1\}\}$. (This topology is called the *Sierpenski space*.) Prove that (X, Ω) is T_0 but is not T_1.

Exercise 1.4.3. Let X be any infinite set endowed with the cofinite topology. Show that this topological space is T_1 but is not T_2.

Exercise 1.4.4. Prove that any T_1 topological space is also T_0, and prove that anyT_2 topological space is also T_1.

Theorem 1.4.5. *If (X, Ω) is any topological space, then the following statements are equivalent:*

1. *The space (X, Ω) is T_1.*
2. *We have $\{x\} \in \kappa(\Omega)$ for all $x \in X$.*
3. *For each $A \subseteq X$, the intersection of all opens containing A is the set A itself.*

Proof. To see that Claim 1 implies Claim 2, let $x \in X$ and let $U = X - \{x\}$. We need to show that $U \in \Omega$. If $U = \emptyset$, there is nothing to show, so suppose that $y \in U$. Since we are dealing with a T_1 space, there exists a neighborhood N of y such that $x \notin N$. We may assume that N is open. Since $x \notin N$, we may conclude that $N \subseteq U$. Therefore, we know that U can be written as a union of open sets and is therefore open.

To see that Claim 2 implies Claim 3, let $A \subseteq X$ and let $\mathcal{F}$ denote the set of all opens containing the set A. Let $B = \bigcap \mathcal{F}$. Clearly we have $A \subseteq B$. We need to prove that $B \subseteq A$. To this end, let $y \in X$. Now, we are assuming that $\{y\}$ is closed; hence, we know that $U = X - \{y\}$ is open. Therefore, if $y \notin A$, we know that $A \subseteq U$. This, of course, means that $U \in \mathcal{F}$; and, since $y \notin U$, we are forced to conclude $y \notin B$. It follows that $y \in B$ must imply that $y \in A$. Consequently, $B \subseteq A$, as desired.

To prove that Claim 3 implies Claim 1, let $x, y \in X$. We need to find neighborhoods M of x and N of y such that $y \notin M$, and $x \notin N$. Let $\mathcal{F}$ denote the family of all opens containins x, and let $\mathcal{G}$ denote the family of all opens containing y. By assumption, $\{x\} = \bigcap \mathcal{F}$ and $y = \bigcap \mathcal{G}$. This tells us that there must exist an open $M \in \mathcal{F}$ such that $y \notin M$. Likewise, there must exist an open $N \in \mathcal{G}$ such that $x \notin N$. We may therefore conclude that (X, Ω) is T_1, as desired.

□

Exercise 1.4.6. Let (X, Ω) be a topological space and let $A \subseteq X$. If (X, Ω) is T_1, then prove that the set of limit points for A is closed.

Definition 1.4.7. We say that a topological space (X, Ω) is T_3 if, for every $C \in \kappa(\Omega)$ and $x \in X - C$, there exist disjoint opens U and V such that $C \subseteq U$ and $x \in V$.

Exercise 1.4.8. Prove that any topological space which is both T_1 and T_3 is Hausdorff. (Such spaces are called *regular*.)

Exercise 1.4.9. Prove that a topological space (X, Ω) is T_3 if and only if, for all $x \in X$ and neighborhoods N of x, there exists a neighborhood M_N of x such that $\overline{M_N} \subseteq N$.

Exercise 1.4.10. Let Ω be defined on $\mathbb{R}$ as follows: We have $U \in \Omega$ if and only if, for each $x \in U$, there exists a segment (a, b) such that $p \in (a, b)$ and each rational number in (a, b) is a member of U.

1. Show that $(\mathbb{R}, \Omega)$ is a topological space.
2. Show that $(\mathbb{R}, \Omega)$ is Hausdorff.
3. Show that $(\mathbb{R}, \Omega)$ is not T_3. Hint: Let p be rational and consider the set $N = \{x \in \mathbb{Q} : p - r < x < p + r \text{ for some } r \in \mathbb{Q}\}$.

Exercise 1.4.11. Let (X, Ω) be a topological space and let (Y, Ω_Y) be a subspace of (X, Ω). If (X, Ω) is Hausdorff, prove that (Y, Ω_Y) is Hausdorff.

Definition 1.4.12. Let (X, Ω) be a topological space and let $A \subseteq X$. A family $\mathcal{F} \subseteq \Omega$ is said to *cover* A provided $A \subseteq \bigcup\{F \in \mathcal{F}\}$. Any subcollection of $\mathcal{F}$ which also covers A is called a *subcover* of A.

Definition 1.4.13. Let (X, Ω) be a topological space and let $A \subseteq X$. We say that A is *compact* relative to Ω provided every cover of A from Ω contains a finite subcover of A. If X is compact, we say that (X, Ω) is a compact space.

Theorem 1.4.14. *Every closed subset of a segment is compact under the usual topology on* $\mathbb{R}$.

Proof. Let S be a segment and let M be a closed subset of $S = (c, d)$. If M is finite, then any cover of M certainly admits a finite subcover, so suppose that M is infinite. We first note that M must have a leftmost and a rightmost point. To see why, M must have a leftmost point, first observe that since c is to the left of every point in M, Theorem 1.1.4 tells us that either M has a leftmost point, or the set N of all points to the left of every point in M has a rightmost point. If M has no leftmost point, then the rightmost point of N is a limit point for M by (the proof of) Theorem 1.1.32. However, this is impossible since M is assumed to be closed and must therefore contain all of its limit points. Thus, M must have a leftmost point. In similar fashion, it can be argued that M has a rightmost point. Let a be the leftmost point of M and let b be the rightmost point of M.

Let $\mathcal{U}$ be an infinite open cover for M. For each $x \in M$, there must exist $U_x \in \mathcal{U}$ such that $x \in U$; and, since the segments form a basis for the usual topology, there must exist a segment S_x such that $x \in S_x \subseteq U_x$. Let $\mathcal{F} = \{S_x : x \in M\}$. Let T be the pointset defined as follows: A point x is a member of T if and only if

- $x = a$ or
- $x \in M$ is to the right of a but not to the right of b, and the set $[a, x] \cap M$ is covered by a finite subcollection of $\mathcal{F}$.

Our goal will be to prove that $b \in T$. Suppose to the contrary that b is not a member of T. By construction, T is nonempty; and b is a point to the right of every point in T. Hence, we know that either T has a rightmost point, or there exists a leftmost point to the set X of all points to the right of every point in T.

Suppose first that T has a rightmost point, ρ. Clearly $\rho \in [a, b]$. Since $\rho \in T$, we know that there exists a finite subcollection $\mathcal{U}$ of segments in $\mathcal{F}$ that covers $[a, \rho] \cap M$. Certainly there is segment U_1 in $\mathcal{F}$ that contains b. If there were no members of M in the segment (ρ, b), then the collection $\mathcal{U} \cup \{U_1\}$ would form a finite subcover of $[a, b] \cap M$. This would mean that $b \in T$ — contrary to assumption. Hence, $M \cap (\rho, b)$ is nonempty. We know that ρ is a limit point of X. Let $S_1, ..., S_n$ be a finite subcover of $[a, \rho] \cap M$. We know that $\rho \in S_i$ for some $1 \leq i \leq n$ and there must exist $x \in X \cap S_i$. This is impossible, because this would make $S_1, ..., S_n$ a finite subcover for $[a, x] \cap M$.

We must conclude that the set X has a leftmost point, λ. The point λ is a limit point of T by (the dual of) Theorem 1.1.32. Since λ is a limit point of T, we know that λ is a limit point of M. Since M is assumed to be closed, this implies that $\lambda \in M$. Hence, we know that λ is contained in a member S_1 of the cover $\mathcal{F}$. The segment S_1 must contain a point $x \in M$. By assumption, there exists a finite subcollection $\mathcal{V}$ of $\mathcal{F}$ that covers the set $[a, x] \cap M$. Hence, the set $\mathcal{V} \cup \{S_1\}$ is a finite subcollection of $\mathcal{F}$ covering $[a, \lambda] \cap M$. This, however, implies that $\lambda \in T$ — contrary to our choice of λ. We are now forced to conclude that $b \in T$, as desired.

We have now extracted a finite subcover $\mathcal{F}'$ from the set $\mathcal{F}$; we may use this to identify a finite subcover in $\mathcal{U}$. Indeed, for each $S_x \in \mathcal{F}'$, simply choose exactly one $U_x \in \mathcal{U}$ such that $S_x \subseteq U_x$.

□

Exercise 1.4.15. Let (X, Ω) be a topological space and let $A \subseteq X$ be compact. Show that every subset of A that is closed relative to Ω is also compact.

Exercise 1.4.16. A subset M of $\mathbb{R}$ is *bounded* provided there exists a segment S such that $M \subseteq S$. Find an example of an unbounded closed subset of $\mathbb{R}$ which is not compact.

Exercise 1.4.17. Let (X, Ω) be a Hausdorff space. Show that every compact subset of X is closed relative to Ω.

Exercise 1.4.18. Consider $\mathbb{R}$ under the usual topology.

1. Prove that $\mathbb{R}$ is Hausdorff.

2. Prove that a subset M of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Exercise 1.4.19. Let (X, Ω) and (Y, Θ) be any topological spaces and let $f : X \longrightarrow Y$ be continuous relative to Ω and Θ. If $A \subseteq X$ is compact relative to Ω, prove that $f(A)$ is compact relative to Θ.

Exercise 1.4.20. Consider $\mathbb{R}$ under the usual topology, and let $M \subseteq \mathbb{R}$ be compact under the subspace topology. If $f : \mathbb{M} \longrightarrow \mathbb{R}$ is continuous, prove that f must have a largest and smallest value. (This is called the *Extreme Value Theorem.*)

Theorem 1.4.21. *Let (X, Ω) be any compact topological space and let (Y, Θ) be a Hausdorff space. If $f : X \longrightarrow Y$ is a bijection and continuous relative to Ω and Θ, then $f^{-1} : Y \longrightarrow X$ is continuous relative to Θ and Ω.*

Exercise 1.4.22. Prove Theorem 1.4.21.

Definition 1.4.23. A family of sets is said to have the *finite intersection property* (FIP) provided the intersection of each of its nonempty finite subfamilies is nonempty.

Theorem 1.4.24. *Let (X, Ω) be a topological space and let $A \subseteq X$. If the set A is compact relative to Ω, then the intersection of any family of closed subsets of A having FIP is nonempty. If A is closed relative to Ω, then the converse is true.*

Proof. Suppose that A is compact, and let $\mathcal{F}$ be a family of closed subsets of A having FIP. We want to prove that $\bigcap \mathcal{F}$ is nonempty. Suppose by way of contradiction that this is not the case. It follows by DeMorgan's Laws that $X = \bigcup \{X - F : F \in \mathcal{F}\}$. Now, the set $\mathcal{U} = \{X - F : F \in \mathcal{F}\}$ forms an open cover for A; hence, we know there exists a finite subcollection $F_1, ...F_n$ such that $\{X - F_1, ...X - F_n\}$ forms a cover for A. However, this coupled with DeMorgan's Laws then implies that $F_1 \cap ... \cap F_n$ is empty — contrary to assumption. Hence, we are forced to conclude that $\bigcap \mathcal{F}$ is nonempty.

Conversely, suppose A is closed and suppose that any family of closed subsets of A having FIP has nonempty intersection. We want to prove that A is compact. Suppose by way of contradiction that A is not compact. Then there exists a family $\mathcal{U}$ of opens that covers A but which admits no finite subcover. Clearly we may assume that, for all $U \in \mathcal{U}$, we have $U \cap A \neq \emptyset$ and $(X - U) \cap A \neq \emptyset$. Consider the set

$$\mathcal{F} = \{(X - U) \cap A : U \in \mathcal{U}\}$$

Note that $\mathcal{F}$ is a family of closed subsets of A. Since $\mathcal{U}$ covers A, we know that $\bigcap \mathcal{F} = \emptyset$. However, since no finite subfamily of $\mathcal{U}$ covers A, it follows that every finite subfamily of $\mathcal{F}$ has nonempty intersection — contrary to assumption.

□

Exercise 1.4.25. Prove that every compact Hausdorff space is regular.

Exercise 1.4.26. Let $\mathcal{B}$ denote the empty set, along with all pairs $\{2n, 2n-1\}$, where n is a positive integer.

1. Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}^+$.
2. Show that the topology generated by $\mathcal{B}$ is not T_0, T_1, or T_2.
3. Show that the topology generated by $\mathcal{B}$ is T_3.
4. If A is any subset of $\mathbb{Z}^+$ containing $2n$, prove that $2n - 1$ is a limit point of A.

1.5 First and Second Countablility

Definition 1.5.1. Let (X, Ω) be a topological space. A subset A of X is *countably compact* relative to Ω provided every countable cover of A in Ω contains a finite subcover.

Theorem 1.5.2. *If A is countably compact, then every infinite subset of A has a limit point in A. The converse is true if the space is T_1.*

Proof. We will prove the contrapositive of both implications. First, suppose that there exists an infinite subset V of A that has no limit point in A. Without loss of generality, we may assume that V is countably infinite. Let $y \in V$. Since y cannot be a limit point of V, there exists an open U_y that contains y but contains no other member of V. Let $\mathcal{U} = \{U_y : y \in V\}$. We know that the collection $\mathcal{U}$ is an open cover for V; furthermore, no member of $\mathcal{U}$ can be deleted. Now suppose $x \in A-V$. Again, since x is not a limit point for V, there exists an open T_x containing x that contains no member of V. Let $T = \bigcup\{T_x : x \in A-V\}$. Clearly $T \cap V = \emptyset$. The collection $\mathcal{U} \cup \{T\}$ is a countable open cover of the set A which admits no finite subcover; hence we know that A is not countably compact.

On the other hand, suppose (X, Ω) is a T_1 space and suppose that A is not countably compact. It follows that there exists a countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of A that admits no finite subcover. We may assume that $\mathcal{U}$ contains no redundant members; that is, we may assume that $U_j \cap A \neq U_k \cap A$ for $j \neq k$. Consequently, for each natural number n, we know that there exists an element $x_n \in A$ such that $x_n \in U_n$ but $x_n \notin U_1 \cup ... \cup U_{n-1}$. Suppose by way of contradiction that this sequence has a limit point $y \in A$. There must exist a natural number n such that $y \in U_n$. Since (X, Ω) is T_1, we know that, for each natural number n and each $1 \leq k \leq n$, there exists an open $U(y, k)$ that contains y but does not contain x_k. Consider the set

$$U = U_n \cap U(y, 1) \cap ... \cap U(y, n)$$

This is an open that contains y but does not contain $x_1, ..., x_n$. Consequently, we know that U must contain x_j for some $j > n$. However, this is impossible, since this would imply that $x_j \in U_n$ — contrary to our choice of x_j. Thus, we are forced to conclude that this sequence has no limit point in A.

□

Exercise 1.5.3. Use Exercise 1.4.26 to show that the converse of Theorem 1.5.2 is false if the T_1 assumption is dropped.

Exercise 1.5.4. Let (X, Ω) be a topological space and let $A \subseteq X$ be countably compact. Prove that every subset of A that is closed relative to Ω is also countably compact.

Definition 1.5.5. Let (X, Ω) be a topological space. We say that Ω is *second-countable* provided Ω has a countable basis.

Theorem 1.5.6. *Let (X, Ω) be a second-countable topological space. If A is an uncountable subset of X, then some point of A is a limit point for A relative to Ω.*

Exercise 1.5.7. Prove Theorem 1.5.6. Hint: Prove the contrapositive.

Exercise 1.5.8. Show that the real numbers under the discrete topology contains uncountable subsets in which no point is a limit point.

Definition 1.5.9. Let (X, Ω) be a topological space. A subset A of X is *dense* in X relative to Ω provided $X = \overline{A}$. We say that (X, Ω) is *separable* provided X contains a countable dense subset.

Exercise 1.5.10. Prove that $\mathbb{R}$ is separable under the usual and the Sorgenfrey topologies. (Consider the rational numbers in both cases.)

Theorem 1.5.11. *Every second-countable topological space is separable.*

Exercise 1.5.12. Prove Theorem 1.5.11.

Theorem 1.5.13. *If (X, Ω) is a second-countable space, then every open cover of X admits a countable subcover.*

Exercise 1.5.14. Prove Theorem 1.5.13.

Exercise 1.5.15. Consider $\mathbb{R}$ under the cofinite topology. Show that $\mathbb{R}$ is separable under this topology but is not second-countable. Hint: Assume the contrary.

Exercise 1.5.16. Suppose (X, Ω) is a topological space, and suppose that $U \subseteq X$ is an open subspace of X. If D is a dense subset of X, prove that $D \cap U$ is dense in U.

Definition 1.5.17. Let (X, Ω) be a topological space and let $x \in X$. The *neighborhood system* for x is the collection $\mathcal{N}_x$ of all neighborhoods relative to Ω that contain x.

Definition 1.5.18. Let (X, Ω) be a topological space and let $x \in X$. A *neighborhood basis* for x is a family $\mathcal{B}_x$ of neighborhoods of x with the property that every member of $\mathcal{N}_x$ contains a member of $\mathcal{B}_x$. If every point of X has a countable neighborhood basis, then we say that (X, Ω) is *first-countable.*

Exercise 1.5.19. Show that every second-countable topological space is also first-countable.

Definition 1.5.20. Let (X, Ω) be a topological space, let $A \subseteq X$, and let $s : \mathbb{N} \longrightarrow X$ be a sequence. We say that s is *frequently in* A if for each $n \in \mathbb{N}$, there exist $m > n$ such that $s_m \in A$. We say that s is *eventually in* A if there exist $n \in \mathbb{N}$ such that $s_m \in A$ for all $m > n$.

Definition 1.5.21. Let (X, Ω) be a topological space and let $s : \mathbb{N} \longrightarrow X$ be a sequence. We say that $x \in X$ is a *cluster point* for s (relative to Ω) provided s is frequently in every neighborhood of x. We say that *s converges to x* (relative to Ω) provided s is eventually in every neighborhood of x.

Let (X, Ω) be a topological space. If a sequence s in X converges to some $x \in X$, then we say that x is a *limit* for s (relative to Ω). Compare this definition of convergence with the one you normally see in calculus. Notice that once again, we have captured the essence of "getting close" to a point without using the notion of distance. Truth be known, we could define convergence entirely in terms of opens, but use of the more general neighborhoods makes proofs easier.

Theorem 1.5.22. *Let (X, Ω) be a first-countable topological space, let $x \in X$ and let $A \subseteq X$. The point x is a limit point of A if and only if there is a sequence in $A - \{x\}$ that converges to x.*

Proof. Suppose there exists a sequence $s : \mathbb{N} \longrightarrow A - \{x\}$ that converges to x. If N is any neighborhood of x, it follows that s is eventually in N. Consequently, every neighborhood of x contains a member of A distinct from x, and we may conclude that x is a limit point for A.

Conversely, suppose that x is a limit point for A. It follows that every neighborhood of x contains an element of $A-\{x\}$. Since (X, Ω) is first-countable, there exists a sequence $U : \mathbb{N} \longrightarrow \Omega$ which serves as a neighborhood basis for x.

We may assume that the sequence U is descending. (Indeed, if this is not the case, we can simply let $V_n = U_1 \cap ... \cap U_n$ for each natural number n and replace the sequence U by the new sequence V.) Now, for each natural number n, the open U_n is a neighborhood of x and hence must contain a member of $A-\{x\}$. Pick such an element and call it s_n. Since the sequence U is descending, it follows that the sequence s so constructed is eventually in every member of U. Now, since U serves as a neighborhood basis for x, it follows that every neighborhood N of x contains a member U_n of U (and therefore contains U_m for all $m > n$). Consequently, the sequence s is eventually in the neighborhood N; and we may conclude that s converges to x.

□

Theorem 1.5.23. *Let (X, Ω) be a first-countable topological space. A subset A of X is open relative to Ω if and only if every sequence in X converging to a member of A is eventually in A.*

Exercise 1.5.24. Prove Theorem 1.5.23.

Theorem 1.5.25. *Let (X, Ω) be a first-countable topological space and let s be a sequence in X. If $x \in X$ is a cluster point of s, then some subsequence of s converges to x (relative to Ω).*

Proof. Let $s : \mathbb{N} \longrightarrow X$ be a sequence and suppose that x is a cluster point for s. Let $U : \mathbb{N} \longrightarrow \Omega$ be a countable neighborhood basis for x. We may assume that the sequence U is descending. Since x is a cluster point for s, we know that s is frequently in each member of U. Construct a new sequence $t : \mathbb{N} \longrightarrow X$ recursively as follows:

1. Let t_1 be the first member of s that is contained in U_1.
2. For each natural number $n > 1$, let j_n be the smallest natural number such that $n < j_n$ and $s_{j_n} \in U_n - \{t_1, ..., t_{n-1}\}$, and let $t_n = s_{j_n}$.

By construction, the new sequence t is a subsequence of s. Moreover, since U is descending, we know by construction that t is eventually in every member of U. Since U is a neighborhood basis for x, it follows that t is eventually in every neighborhood of x. Consequently, t converges to x, as desired.

□

Theorem 1.5.26. *Let (X, Ω) and (Y, Θ) be first-countable topological spaces and let $p \in X$. A function $f : X \longrightarrow Y$ is continuous at p relative to Ω and Θ if and only if the sequence $f(s)$ in Y converges to $f(p)$ for every sequence s in X that converges to p.*

Exercise 1.5.27. Prove Theorem 1.5.26.

Exercise 1.5.28. If s is a strictly increasing sequence in $\mathbb{Z}^+$ show that s converges to every positive integer relative to the cofinite topology. (Hence, a sequence can have more than one limit in some topologies.)

Theorem 1.5.29. *A first-countable topological space is Hausdorff if and only if every sequence in the space converges to at most one point in the space.*

Exercise 1.5.30. Prove Theorem 1.5.29.

The following result is sometimes called *Lindelöf's Theorem.* Its proof is straightforward, but the result has important consequences.

Theorem 1.5.31. *Let (X, Ω) be a second-countable space. If $\mathcal{G}$ is any family from Ω, then there exists a countable $\mathcal{U} \subseteq \mathcal{G}$ such that $\bigcup \mathcal{U} = \bigcup \mathcal{G}$.*

Proof. Let $\mathcal{B}$ be a countable basis for Ω and suppose $\mathcal{G} \subseteq \Omega$. Notice that every member of $\mathcal{G}$ must contain at least one member of $\mathcal{B}$, since $\mathcal{B}$ is a basis for Ω. Consider the family

$$\mathcal{C} = \{B \in \mathcal{B} : B \subseteq G \text{ for some } G \in \mathcal{G}\}$$

Of course, $\mathcal{C}$ is a countable set. For each $B \in \mathcal{C}$, let $G_B \in \mathcal{G}$ such that $B \subseteq G_B$ and let $\mathcal{U} = \{G_B : B \in \mathcal{C}\}$. By construction, $\mathcal{U}$ is a countable subset of $\mathcal{G}$; consequently, we know that $\bigcup \mathcal{U} \subseteq \bigcup \mathcal{G}$. For the reverse inclusion, suppose $x \in \bigcup \mathcal{G}$. There must exist $G \in \mathcal{G}$ such that $x \in G$. There must exist $B \in \mathcal{B}$ such that $x \in B \subseteq G$. However, since $B \subseteq G$ for some $G \in \mathcal{G}$, we know that $B \in \mathcal{C}$. Hence, we also know that $x \in G_B$. It follows that $x \in \bigcup \mathcal{U}$; and we may conclude that $\bigcup \mathcal{G} \subseteq \bigcup \mathcal{U}$.

□

Exercise 1.5.32. If (X, Ω) is a second-countable topological space, then every basis for Ω contains a countable subset which is also a basis for Ω.

Definition 1.5.33. We say that a topological space (X, Ω) is a *Lindelöf space* provided every open cover of X admits a countable subcover.

Notice that Lindelöf's Theorem tells us that every second-countable topological space is a Lindelöf space. The converse of this statement is false, however.

Theorem 1.5.34. *The Sorgenfrey Line is a Lindelöf space which is not second-countable.*

Proof. To see that the Sorgenfrey Line $(\mathbb{R}, \Omega_S)$ is Lindelöf, suppose that $\mathcal{U}$ is an open cover for $\mathbb{R}$ relative to this topology. Now, if $a \in \mathbb{R}$, then we know that $a \in U$ for some $U \in \mathcal{U}$. Furthermore, since we are working in the Sorgenfrey space, there must exist a real number $b_a > a$ such that $[a, b_a) \subseteq U$. We will work with the collection

$$\mathcal{V} = \{[a, b_a) : a \in \mathbb{R}, a < b, \text{ and}[a, b) \subseteq U \text{ for some} U \in \mathcal{U}\}$$

Our goal will be to extract a countable subcover from the family $\mathcal{V}$. Let $\mathcal{M} = \{(a, b_a) : a \in \mathbb{R}\}$ and let K be the set of all real numbers not contained in any member of $\mathcal{M}$. Suppose by way of contradiction that K is uncountable. The contrapositive of Exercise 1.1.37 tells us there exist $y \in K$ such that, for all $\epsilon > 0$, we have $(y, y+\epsilon) \cap K \neq \emptyset$. In particular, we know that $(y, b_y) \cap K \neq \emptyset$. Consequently, there exist $z \in (y, b_y) \cap K$. However, if $z \in (y, b_y)$, then z is covered by $\mathcal{M}$. This implies that $z \notin K$ — a contradiction. We must conclude that K is countable.

Now, $\mathbb{R}$ is certainly second-countable under the usual topology. Clearly, the set $\mathcal{M}$ forms an open cover of $\mathbb{R} - K$ under the usual topology; by Lindelf̈'s

Theorem, the family $\mathcal{M}$ therefore admits a countable subcover $\mathcal{M}'$ for $\mathbb{R} - K$. Consider the family

$$\mathcal{N} = \{[a, b_a) : (a, b_a) \in \mathcal{M}'\} \cup \{[y, b_y) : y \in K\}$$

The family $\mathcal{N}$ is the union of two countable sets and is therefore countable. Furthermore, $\mathcal{N} \subseteq \mathcal{V}$ and certainly forms an open cover for $\mathbb{R}$ under the Sorgenfrey topology. For each $Y_n \in \mathcal{N}$, let U_n be one member of $\mathcal{U}$ such that $Y_n \subseteq U$. The family $\mathcal{U}' = \{U_n : n \in \mathbb{N}\}$ is a countable subcover of $\mathbb{R}$.

It remains to prove that $\mathbb{R}$ is not second-countable under the Sorgenfrey topology. Suppose by way of contradiction that the Sorgenfrey line is second-countable. Since $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is a basis for this space, Lindelöf's Theorem tells us that $\mathcal{B}$ contains a countable set which is also a basis. Let $\mathcal{C} = \{[a_n, b_n) : n \in \mathbb{N}\}$ be such a basis. Since $\mathbb{R}$ is uncountable, there exist real numbers x such that $x \neq a_n$ for any natural number n. Consider the set $[x, x+1)$. There must exist some natural number n such that $x \in [a_n, b_n) \subseteq [x, x + 1)$. However, this is clearly impossible, since this would force $x = a_n$.

□

Exercise 1.5.35. Prove that every closed subspace of a Lindelöf space is also a Lindelöf space.

Theorem 1.5.36. *If (X, Ω) is a second-countable topological space, then the following statements are equivalent:*

1. *A subset A of X is compact relative to Ω.*
2. *Every sequence in A contains a subsequence that converges to some point in X.*

Exercise 1.5.37. Prove Theorem 1.5.36.

1.6 Product Spaces

We now turn attention to ways of constructing topologies on products of sets. In all that follows, we will assume that for any set A, we have $A \times \emptyset = \emptyset = \emptyset \times A$. This notational convention will help to simplify some of the arguments to follow.

Exercise 1.6.1. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty families of sets. Prove the following statements are true.

1. We have $\bigcap \mathcal{A} \times \bigcap \mathcal{B} = \bigcap \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$.
2. We have $\bigcup \mathcal{A} \times \bigcup \mathcal{B} = \bigcup \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

Exercise 1.6.2. Suppose that $A_1, ..., A_n$ and $B_1, ..., B_n$ are sets. Prove that

$$(A_1 \times B_1) \cap ... \cap (A_n \times B_n) = (A_1 \cap ... \cap A_n) \times (B_1 \cap ... \cap B_n)$$

Exercise 1.6.3. Show by example that $(A \cup C) \times (B \cup D) \neq (A \times B) \cup (C \times D)$ in general.

Exercise 1.6.4. Suppose that (X, Ω) and (Y, Θ) are topological spaces. Let $\Omega \times \Theta$ denote the collection of sets of the form $U \times V$, where $U \in \Omega$ and $V \in \Theta$. Prove that $\Omega \times \Theta$ generates a topology on $X \ \times Y$.

Let (X, Ω) and (Y, Θ) be topological spaces. We will let Ω_{XY} denote the topology generated on $X \times Y$ by the family $\Omega \times \Theta$ described above and call $(X \times Y, \Omega_{XY})$ the *product* topology on X and Y (relative to Ω and Θ). In light of Exercise 1.6.3, the product topology on $X \times Y$ is more complicated than might initially be expected — open sets in the product topology do not correspond directly to products of opens from the respective topologies, at least not in a straightforward manner. They are built up from products by taking unions; and we may assume that we are taking unions of products from the bases for Ω and Θ, as the next result proves.

Exercise 1.6.5. Let (X, Ω) and (Y, Θ) be topological spaces and suppose that $\mathcal{B}_\Omega$ and $\mathcal{C}_\Theta$ are bases for Ω and Θ, respectively. Let

$$\mathcal{B}_\Omega \times \mathcal{B}_\Theta = \{B \times C : B \in \mathcal{B}_\Omega \text{ and } C \in \mathcal{C}_\Theta\}$$

Prove that $\mathcal{B}_\Omega \times \mathcal{B}_\Theta$ is a basis for the product topology on X and Y (relative to Ω and Θ).

Suppose we endow $\mathbb{R}$ with the usual topology. The resulting product topology on $\mathbb{R} \times \mathbb{R}$ is called the usual product topology. What do the basic opens of this topology look like? We know that a basis for $\mathbb{R}$ under the usual topology is simply a the family of all segments (a, b) along with $\emptyset$. If we imagine $\mathbb{R} \times \mathbb{R}$ to be the standard cartesian plane, then a basis for the usual product topology would be the family of "rectangles" in the plane that do not contain their boundaries. That is, a basic open is either $\emptyset$ or a set of the form

$$B = \{(x, y) : a < x < b \text{ and } c < y < d \text{ for some } a, b, c, d \in \mathbb{R}\}$$

Definition 1.6.6. Let$\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ be defined by $\pi_X[(x, y)] = x$ and $\pi_Y[(x, y)] = y$. We call these functions the *projection maps* of $X \times Y$.

Exercise 1.6.7. Let (X, Ω), (Y, Θ) be topological spaces and let $(X \times Y, \Omega_{XY})$ be the product topology relative to Ω and Θ. Prove that π_X and π_Y are continuous relative to the appropriate topologies.

If (X, Ω) and (Y, Θ) are topological spaces, then it stands to reason that the product spaces $(X \times Y, \Omega_{XY})$ and $(Y \times X, \Omega_{YX})$ should be essentially the same. The following definition provides a way to show this is indeed the case.

Definition 1.6.8. Let (X, Ω) and (Y, Θ) be topological spaces. A mapping $f : X \longrightarrow Y$ is called a *homeomorphism* provided

1. the mapping f is a bijection,
2. the mapping f is continuous relative to Ω and Θ,
3. the mapping f^{-1} is continuous relative to Θ and Ω.

We say the topological spaces are `homeomorphic` when this is the case.

The notion of "homeomorphism" described above provides us with a way to determine when two topological spaces are essentially the same. This notion does for topological spaces what isomorphisms do for groups.

Exercise 1.6.9. Suppose that (X, Ω) and (Y, Θ) are homeomorphic topological spaces and suppose that $f : X \longrightarrow Y$ is a homeomorphism. Show that $U \in \Omega$ if and only if $f(U) \in \Theta$.

Exercise 1.6.10. Suppose that (X, Ω) and (Y, Θ) are homeomorphic topological spaces and suppose that $f : X \longrightarrow Y$ is a homeomorphism. Let $\mathcal{B}$ be a collection of subsets of X. Prove that $\mathcal{B}$ is a basis for Ω if and only if $f(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$ is a basis for Θ.

Exercise 1.6.11. Suppose that (X, Ω) is any topological space. For any point a let $\Theta = \{\emptyset, \{a\}\}$. Prove that $X \times \{a\}$ under the product topology relative to Ω and Θ is homeomorphic to (X, Ω).

Exercise 1.6.12. Let (X, Ω) and (Y, Θ) be any topological spaces. Prove that $(X \times Y, \Omega_{XY})$ is homeomorphic to $(Y \times X, \Omega_{YX})$.

Exercise 1.6.13. Let a and b are any real numbers and let $f : [0, 1] \longrightarrow [a, b]$ be defined by $f(x) = a + (b - a)x$. Using this function, prove that the interval $[0, 1]$ is homeomorphic to $[a, b]$ (relative to the usual subspace topologies for both).

Exercise 1.6.14. Prove that the open segment $(0, 1)$ is homeomorphic to the open segment (a, b) for any real numbers $a < b$ (relative to the usual subspace topologies for both).

Exercise 1.6.15. Prove that the open segment $(0,1)$ (under the usual subspace topology) is homeomorphic to $\mathbb{R}$ under the usual topology. Consider the function

$$f(x) = \frac{1}{x} + \frac{1}{x-1}$$

The product topology described above can be extended to any finite collection of topological spaces in a straightforward manner. Suppose (X_1, Ω_1), ..., (X_n, Ω_n) are topological spaces. Let

$$\prod_{j=1}^{n} \Omega_n = \{U_1 \times ... \times U_n : U_j \in \Omega_j\}$$

The product topology on $X_1 \times ... \times X_n$ is the topology generated by $\prod_{j=1}^{n} \Omega_n$. As in the two-set case, a basis for this topology can be constructed by taking the cartesian product of bases for the respective factor topologies. (The product topology is sometimes called the *box topology* on $X_1 \times ... \times X_n$, since basic opens are "boxes" whose "faces" are basic opens from Ω_1, ..., Ω_n.)

Exercise 1.6.16. Let (X_1, Ω_1), ..., (X_n, Ω_n) be topological spaces. Prove that $X_1 \times ... \times X_n$ is homeomorphic to $(X_1 \times ... \times X_{n-1}) \times X_n$ under the product topologies on both.

In light of the previous exercise, a simple induction argument based on Exercise 1.6.12 proves that, up to homeomorphism, the order of the factors does not matter in the product topology on (X_1, Ω_1), ..., (X_n, Ω_n).

Exercise 1.6.17. Let (X, Ω) and (Y, Θ) be topological spaces. If $A \subseteq Y$ is compact (relative to Θ) and $x \in X$, show that $\{x\} \times Y$ is compact relative to the product topology on $X \times Y$.

Exercise 1.6.18. Let (X, Ω) and (Y, Θ) be topological spaces, let $x \in X$ and let $A \subseteq Y$ be compact. If N is any open set of the product topology on $X \times Y$ containing $\{x\} \times A$, prove there is an open W containing x (relative to Ω) such that $W \times A \subseteq N$.

Theorem 1.6.19. *Let (X, Ω) and (Y, Θ) be topological spaces, let $A \subseteq X$ and $B \subseteq Y$. The sets A and B are compact relative to Ω and Θ if and only if $A \times B$ is compact relative to the product topology on $X \times Y$.*

Proof. First, suppose that $A \times B$ is compact in the product topology. Let $\mathcal{U}$ be an open cover for A and let $\mathcal{V}$ be an open cover for B. It follows that the collection $\mathcal{F} = \{U \times V : U \in \mathcal{U}, V \in \mathcal{V}\}$ is an open cover for $A \times B$. Consequently, we know that $\mathcal{F}$ contains a finite subcollection $\{U_1 \times V_1, ...U_n \times V_n\}$ which also covers $A \times B$. The collections $U_1, ..., U_n \in \mathcal{U}$ and $V_1, ..., V_n \in \mathcal{V}$ therefore cover A and B, respectively.

Conversely, suppose that A and B are compact relative to Ω and Θ, respectively. Let $\mathcal{U}$ be any open cover of $A \times B$ in the product topology. For each $a \in A$, the set $\{a\} \times B$ is compact relative to the product topology; consequently, there exists a finite subcollection $\{U(1, a), ..., U(n, a)$ of $\mathcal{U}$ that covers $\{a\} \times B$. Now, the set $N_a = U(1, a) \cup ... \cup U(n, x)$ is an open that contains $\{a\} \times B$; hence, by the previous exercise, there exists an open W_a containing a such that $W \times B \subseteq N$. Since N is the union of a finite subcollection of $\mathcal{U}$, it follows that $W_a \times B$ is covered by this subcollection.

Now, the set $\mathcal{W} = \{W_a : a \in A\}$ is an open cover of A in Ω; hence we know that there exists a finite subcollection $W_1, ..., W_m$ which covers A. It follows that $A \times B \subseteq (W_1 \times B \cup ... \cup (W_m \times B)$. Since each of the sets $W_j \times B$ can be covered by a finite subcollection of $\mathcal{U}$, we now see that $A \times B$ can also be covered by a finite subcollection of $\mathcal{U}$ (just take the union of the subcollections for each $1 \leq j \leq m$). Hence, $A \times B$ is compact, as desired.

$\square$

A simple induction argument now tells us that the previous theorem can be extended to finite products. In particular, if (X_1, Ω_1), ..., (X_n, Ω_n) are topological spaces, and if $A_j \subseteq X_j$ is compact relative to Ω_j for $1 \leq j \leq n$, then $A_1 \times ... \times A_n$ is compact in $X_1 \times ... \times X_n$ relative to the product topology.

We conclude this section by extending the notion of product topology to *arbitrary* collections of topological spaces. This process is more subtle than you might initially think, especially since it is not obvious how we would even go about defining the product of an infinite family of sets.

Definition 1.6.20. A set X is said to be *indexed* by a set I provided there exists a bijection $f : I \longrightarrow X$. We call f an *indexing* of X by the set I, and we typically let x_i denote the element $f(i)$ for each $i \in I$.

Definition 1.6.21. Let $\mathcal{X} = \{X_i : i \in I\}$ be a family of sets indexed by the set I. We define the *direct product* of these sets to be the set of all mappings $t : I \longrightarrow \bigcup \mathcal{X}$ such that $t(i) \in X_i$ for each $i \in I$. We use the symbol $\prod\{X_i : i \in I\}$ to denote this set of mappings. The elements $t(i)$ are called the *coordinates* of the mapping t, and the sets X_i are called the `factors` of the product.

If $\mathcal{X} = \{X_1, ..., X_n\}$ is a finite collection of sets, then we can let the index set be the cardinal number $N = \{1, ..., n\}$. In this case, a member t of $\prod\{X_i : i \in I\}$ can be identified with an ordered n-tuple $(x_1, ..., x_n)$, where each $x_i = t(i)$. Consequently, for finite families of sets, we identify the direct product with the standard cartesian product. The previous discussion can be extended to countably infinite direct products as well. Indeed, the set $\mathbb{Z}^\infty$ of all sequences of integers can be thought of as the direct product of countably many copies of the set $\mathbb{Z}$ of integers.

Let $\{X_i : i \in I\}$ be a family of sets indexed by a set I. As with finite cartesian products, if $X_i = \emptyset$ for some $i \in I$, we will set $\prod\{X_i : i \in I\} = \emptyset$.

Exercise 1.6.22. Let I be an index set and let $\mathcal{X} = \{X_i : i \in I\}$ and $\mathcal{Y} = \{Y_i : i \in I\}$ be families of sets indexed by I. Prove that

$$\prod \mathcal{X} \cap \prod \mathcal{Y} = \prod\{X_i \cap Y_i : i \in I\}$$

Definition 1.6.23. Let I be an index set and let $\mathcal{X} = \{X_i : i \in I\}$ be a family of sets indexed by I. For each $i \in I$, let $\pi_i : \prod\{X_i : i \in I\} \longrightarrow X_i$ be defined by $\pi_i(t) = t(i)$. We call these functions the *projection maps* for the product.

Exercise 1.6.24. Let $\mathcal{X} = \{X_i : i \in I\}$ be a family of sets indexed by the set I. If $A \subseteq X_i$, prove that $\pi_i^{-1}(A) = \prod\{U_j : j \in I\}$, where $U_i = A$ and $U_j = X_j$ otherwise.

Exercise 1.6.25. Let $\mathcal{X} = \{X_i : i \in I\}$ be a family of sets indexed by the set I. Let $j_1, ..., j_n \in I$ and suppose that $A_{j_k} \subseteq X_{j_k}$ for $1 \leq k \leq n$. Let $B = \prod\{U_i : i \in I\}$, where $U_{j_k} = A_{j_k}$ and $U_i = X_i$ if $i \neq j_k$. Prove that $B = \bigcap\{\pi_{j_k}^{-1}(A_{j_k}) : 1 \leq k \leq n\}$.

Exercise 1.6.26. Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of topological spaces indexed by I. Let $\mathtt{B}$ denote the family of all products of collections $\mathcal{U} = \{U_i : U_i \in \Omega_i\}$. Show that $\mathtt{B}$ generates a topology on $\prod\{X_i : i \in I\}$.

We will let Ω_b denote the topology generated by the family $\mathtt{B}$ in the previous exercise. Note that this topology is a natural extension of the product topology we defined for finite families of topological spaces. Consequently, we call Ω_b the "box" topology on the family $\{X_i : i \in I\}$ relative to the topologies $\{\Omega_i : i \in I\}$. You might wonder why we don't simply call this the "product" topology like we did for finite collections. The following exercises explain why.

Exercise 1.6.27. Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of topological spaces indexed by I. Let $\mathtt{P}$ denote the family of all products of collections

$$\mathcal{U} = \{U_i : U_i \in \Omega_i \text{ and all but finitely many } U_i = X_i\}$$

Show that the set $\mathtt{P}$ generates a topology on $\prod\{X_i : i \in I\}$. We will let Ω_p denote this topology.

Exercise 1.6.28. Let $\mathcal{T} = \{(X_1, \Omega_1), ..., (X_n, \Omega_n)\}$. Prove that $\Omega_b = \Omega_p$ on $\prod\{X_1, ..., X_n\}$.

Exercise 1.6.29. Let $\{(\mathbb{R}_i, \Omega_i) : i \in \mathbb{N}\}$ be an indexed family of copies of the real line under the usual topology and let $\mathbb{R}^\infty = \prod\{\mathbb{R}_i : i \in \mathbb{N}\}$. Show that Ω_p is a proper subset of Ω_b on $\mathbb{R}^\infty$. (Hence, the box topology is not the same as the topology Ω_p in this case.

Exercise 1.6.30. Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of topological spaces indexed by I. Show that the projection maps are continuous under both the box and product topologies on $\prod\{X_i : i \in I\}$.

In fact, Exercises 1.6.24 and 1.6.25 tell us that the topology Ω_p is the coarsest (that is the smallest) topology in which the projection maps are continuous. For this reason, we opt to call Ω_p the "product" topology for a family of topological spaces.

Exercise 1.6.31. Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of topological spaces indexed by I. Let

$$\mathcal{P} = \{\pi_i^{-1}(U) : i \in I \text{ and } U \in \Omega_i\}$$

Let $\mathcal{P}'$ denote the set of all intersections of finite nonempty subcollections of $\mathcal{P}$. Show that $\mathcal{P}'$ is a basis for the product topology.

Exercise 1.6.32. Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of topological spaces indexed by I and let $(\prod\{X_i : i \in I), \Omega_p)$ be the product topology. Let (A, Θ) be any topological space, suppose that $\mathcal{F}$ is a family of functions $f_i : A \longrightarrow X_i$. Prove that the function $f : A \longrightarrow \prod\{X_i : i \in I\}$ defined by $\pi_i(f(a)) = f_i(a)$ is continuous relative to Θ and Ω_p if and only if each f_i is continuous relative to Θ and Ω_i.

Exercise 1.6.33. For each $i \in \mathbb{Z}^+$, let $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ be the identity map $f_i(x) = x$.

1. Show that each f_i is continuous relative to the usual topology on $\mathbb{R}$.
2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^\infty$ be defined by $\pi_i(f(x)) = f_i(x) = x$ for all $i \in \mathbb{N}$. Show that f is not continuous relative to the box topology. Hint: Let $U = \prod\{(-1/n, 1/n) : n \in \mathbb{Z}^+\}$ and prove that $f^{-1}(U)$ is not open.

Exercise 1.6.34. Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of topological spaces indexed by I. If each (X_i, Ω_i) is Hausdorff, prove that the product topology $(\prod \mathcal{T}, \Omega_p)$ is Hausdorff.

Exercise 1.6.35. Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of pairwise disjoint topological spaces indexed by I.

1. Prove that if $(\prod \mathcal{T}, \Omega_p)$ is first-countable, then all but a countable number of the spaces (X_i, Ω_i) are indiscrete.

2. If, in addition, all the spaces are first-countable, prove that the converse of Claim (1) is true.

Exercise 1.6.36. Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of topological spaces indexed by I. If $A_i \subseteq X_i$ is closed relative to Ω_i, then $\prod\{A_i : i \in I\}$ is closed relative to the product topology.

Having spent some extra time exploring what it means to construct products of arbitrary families of topological spaces, we will now complete the project begun in Theorem 1.6.19. In particular, we will prove *Tychonoff's Theorem*:

Theorem 1.6.37. *Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of topological spaces indexed by I. If $A_i \subseteq X_i$ is compact and closed relative to Ω_i, then $\prod\{A_i : i \in I\}$ is compact (and closed) in the product topology.*

The proof of Tychonoff's Theorem is not accomplished by appealing directly to the open cover definition of compactness, but rather by exploiting the Finite Intersection Property (FIP) as it applies to compactness (see Theorem 1.4.24). We begin this task with an important concept from order theory.

Definition 1.6.38. Let X be any set and let $\mathcal{S}$ be any collection of subsets of X. We say that a subset $\mathcal{C}$ of $\mathcal{S}$ is a *chain* provided $A, B \in \mathcal{C}$ implies that $A \subseteq B$ or $B \subseteq A$. Such collections are also said to be *totally ordered* or *linearly ordered*.

Definition 1.6.39. Let X be any set and let $\mathcal{S}$ be any collection of subsets of X. A member M of $\mathcal{S}$ is *maximal* in $\mathcal{S}$ if, whenever $B \in \mathcal{S}$ is such that $M \subseteq B$, then $M = B$.

In order theory, the following result is traditionally called *Zorn's Lemma* in honor of Max Zorn, even though it is technically not a lemma and was not proven by Zorn.

Theorem 1.6.40. *Let X be any set and let $\mathcal{S}$ be any collection of subsets of X. The following statements are equivalent:*

1. *If every chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$, then $\mathcal{S}$ has a maximal member.*

2. *If every chain in $\mathcal{S}$ has a least upper bound in $\mathcal{S}$, then $\mathcal{S}$ has a maximal member.*

3. *The collection $\mathcal{S}$ contains a maximal chain.*

Proof.. We first prove that Claim (1) implies Claim (2). To this end, suppose that every chain in $\mathcal{S}$ has a least upper bound in $\mathcal{S}$. Clearly, then, every chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$; and by Claim (1), we know that $\mathcal{S}$ has a maximal member.

We now prove that Claim (2) implies Claim (3). Let $\mathcal{C}$ denote the collection of all chains in $\mathcal{S}$, and let $\mathbb{C}$ be a chain in $\mathcal{C}$. The set $\bigcup \mathbb{C}$ is a chain in $\mathcal{S}$ and hence has a least upper bound, $c \in \mathcal{S}$. It follows that the family $\bigcup \mathbb{C} \cup \{c\}$ is the least upper bound of $\mathbb{C}$ in $\mathcal{C}$. Hence, by Claim (2), $\mathcal{C}$ contains a maximal member; which is, of course, a maximal chain in $\mathcal{S}$.

Finally, we prove that Claim (3) implies Claim (1). Suppose $\mathcal{S}$ contains a maximal chain, C. If every chain in $\mathcal{S}$ has an upper bound, it follows that C has an upper bound, m. First, it is clear that $m \in C$; otherwise, $C \cup \{m\}$ would be a chain properly containing C — contrary to the maximality of C. Second, it is clear that m is the largest element of C — otherwise, there would be members of C properly containing m. Now, it is clear that m is maximal in $\mathcal{S}$. Indeed, if this were not the case, then there would exist $n \in \mathcal{S}$ properly containing m. Since m is the largest member of C, it would follow that $C \cup \{n\}$ would be a chain properly containing C — again contrary to the maximality of C.

□

Exercise 1.6.41. Let X be any set and let $\mathcal{S}$ be any family of subsets from X. Prove that the following statements are equivalent to Zorn's Lemma.

1. If every chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$, then every chain in $\mathcal{S}$ is contained in a maximal chain.

2. If every chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$, then each chain has a maximal upper bound in $\mathcal{S}$.

3. If $\bigcup \mathcal{C} \in \mathcal{S}$ for every chain $\mathcal{C}$ in $\mathcal{S}$, then $\mathcal{S}$ has a maximal member.

4. If a set U is a member of $\mathcal{S}$ if and only if every finite subset of U is a member of $\mathcal{S}$, then every member of $\mathcal{S}$ is contained in a maximal member of $\mathcal{S}$.

Exercise 1.6.42. Let X be any set. Prove that every family $\mathcal{A}$ of subsets of X having the finite intersection property is contained in collection of subsets of X that is maximal with respect to the finite intersection property.

Exercise 1.6.43. Let X be any set and suppose that $\mathcal{M}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Prove the following statements are true.

1. The intersection of any finite nonempty subcollection of $\mathcal{M}$ is a member of $\mathcal{M}$.

2. Any subset of X that is not disjoint with every member of $\mathcal{M}$ is contained in $\mathcal{M}$.

Proof of Tychonoff's Theorem

Let I be an index set and let $\mathcal{T} = \{(X_i, \Omega_i) : i \in I\}$ be a family of topological spaces indexed by I. Let $A_i \subseteq X_i$ be compact and closed relative to Ω_i. We want to prove that $\prod\{A_i : i \in I\}$ is compact in the product topology.

Let $A = \prod\{A_i : i \in I\}$. First, note that by Exercise 1.6.36, A is closed under the product topology. Hence, we may invoke Theorem 1.4.24. It will suffice to prove that if $\mathcal{F}$ is any collection of closed subsets A having FIP, then $\bigcap \mathcal{F}$ is nonempty. To this end, suppose that $\mathcal{F}$ is a family of subsets of A having FIP. It follows by Exercise 1.6.42 that $\mathcal{F}$ is contained in a family $\mathcal{M}$ of subsets of A which is maximal with respect to FIP. Now, the members of $\mathcal{M}$ need not all be closed relative to Ω_p; however, it will still suffice to prove that $B = \bigcap\{\overline{M} : M \in \mathcal{M}\}$ is nonempty, since this is clearly a subset of $\bigcap \mathcal{F}$.

Let $i \in I$ and consider the collection $\mathcal{M}_i = \{\pi_i(M) : M \in \mathcal{M}\}$. Since $\mathcal{M}$ has FIP in $\prod\{X_i : i \in I\}$, it follows that $\mathcal{M}_i$ has FIP in X_i. Consider the collection

$$\mathcal{N}_i = \{\overline{\pi_i(M)} : M \in \mathcal{M}\}$$

Since A_i is closed relative to Ω_i, we know that $\mathcal{N}_i$ is a collection of closed subsets of A_i having FIP. Hence, by the compactness of A_i, we know that $\bigcap \mathcal{N}_i$ is nonempty. Consequently, we know there exist $x_i \in \bigcap \mathcal{N}_i$. Define a mapping $x : I \longrightarrow \bigcup\{X_i : i \in I\}$ by letting $x(i) = x_i$ for each $i \in I$. We will prove that $x \in B$.

It will suffice to prove that $x \in \overline{M}$ for every $M \in \mathcal{M}$. For each $i \in I$, suppose that U_i is an open in Ω_i which contains x_i. By assumption, $x_i \in \overline{\pi_i(M)}$ for all $M \in \mathcal{M}$; hence, we know that $U_i \cap \overline{\pi_i(M)}$ is nonempty for all $M \in \mathcal{M}$. Either $x_i \in \pi_i(M)$, or $x_i \notin \pi_i(M)$. If $x_i \notin \pi_i(M)$, then we know that x_i is in the boundary of $\pi_i(M)$. This implies that U_i must contain members of $\pi_i(M)$ by Exercise 1.2.29. Thus, in either case, we know that $U_i \cap \pi_i(M)$ is nonempty. This implies that $\pi_i^{-1}(U_i) \cap M$ is nonempty for all $M \in \mathcal{M}$.

We now know from Exercise 1.6.43 (2) that $\pi_i^{-1}(U_i) \in \mathcal{M}$. It now follows from Exercise 1.6.43 (1) and Exercise 1.6.31 that every basic open in Ω_p that contains x is also a member of $\mathcal{M}$ (and thus has nonempty intersection with every member of $\mathcal{M}$). Thus, if $M \in \mathcal{M}$, we know that every open in Ω_p containing x has nonempty intersection with M. This tells us that x is either a member of M or is a member of the boundary of M by Exercise 1.2.29; hence $x \in \overline{M}$ as desired.

$\square$

Given any subset X of $\mathbb{R}^\infty$, we will let (X, Ω_p) denote X under the subspace product topology; and if $Y \subseteq \mathbb{R}$, then we will let Y^∞ denote the set $\prod\{Y_j : j \in \mathbb{Z}^\infty\}$, where $Y_j = Y$ for all j.

Theorem 1.6.44. *If a and b are any real numbers, then $([0,1]^\infty, \Omega_p)$ is homeomorphic to $([a,b]^\infty, \Omega_p)$.*

Proof. First, we know by Tychonoff's Theorem that both $([0,1]^\infty, \Omega_p)$ and $([a,b]^\infty, \Omega_p)$ are compact. By Exercise 1.6.34, we know that $([a,b]^\infty, \Omega_p)$ is a Hausdorff space. Consider the mapping $F : [0,1]^\infty \longrightarrow [a,b]^\infty$ defined by

$$\pi_n[F(x)] = a + (b-a)x_n$$

Since each function $f_n(x) = a + (b-a)x_n$ is a homeomorphism, it follows that F is a continuous bijection. By Theorem 1.4.21, we may conclude that F^{-1} is also continuous.

$\square$

1.7 Metric Spaces

Definition 1.7.1. Let X be any set and suppose that $\mu : X \times X \longrightarrow \mathbb{R}$ is a function. We say that μ is a *metric* on X provided the following properties hold:

1. We have $\mu(x,y) \geq 0$ for all $x, y \in X$.
2. We have $\mu(x,y) = 0$ if and only if $x = y$.
3. We have $\mu(x,y) = \mu(y,x)$ for all $x, y \in X$.
4. We have $\mu(x,y) \leq \mu(x,z) + \mu(z,y)$ for all $x, y, z \in X$.

The last condition in the definition of a metric appearing above is called the *triangle inequality*; it is an extension of the familiar triangle inequality for the distance function defined on the real numbers. Indeed, every property mentioned above comes directly from our intuitive understanding of what "distance between points" should be. We can use metrics to define a concept of *distance* between points for a number of topological spaces.

Exercise 1.7.2. Let X be any nonempty set and let $\delta : X \times X \longrightarrow \mathbb{R}$ be defined by $\delta(x,y) = 0$ if $x = y$ and $\delta(x,y) = 1$ if $x \neq y$. Prove that μ is a metric on X. (This is called the *discrete* metric on X.)

Exercise 1.7.3. Show that the mapping $D : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $D(x, y) = (x - y)^2$ fails to be a metric.

Exercise 1.7.4. Let X be any nonempty set equipped with metrics μ and ν. Prove that the mappings $M : X \times X \longrightarrow \mathbb{R}$ and $T : X \times X \longrightarrow \mathbb{R}$ defined by

1. $M(x, y) = \mathtt{max}[\mu(x, y), \nu(x, y)]$
2. $T(x, y) = \mu(x, y) + \nu(x, y)$

are also a metrics.

Exercise 1.7.5. Let x and y be real numbers and recall that the *Euclidean distance* between x and y is given by

$$|x - y| = \begin{cases} x - y & \text{if } x \leq y, \\ y - x & \text{otherwise.} \end{cases}$$

Prove that $\mu(x, y) = |x - y|$ is a metric on $\mathbb{R}$.

Exercise 1.7.6. Prove that the following are metrics on the cartesian product $\mathbb{R}^n$.

$$M(\mathbf{x}, \mathbf{y}) = \mathtt{Max}\{|x_i - y_i| : 1 \leq i \leq n\} \qquad T(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

The function T defined in the previous exercise is often called the "taxicab metric" because it measures distance along such a convoluted path between two points.

No doubt, the most familiar metric on $\mathbb{R}^n$ is the so-called "distance formula" learned in basic algebra: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, let

$$E(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

In formal mathematics circles, this is known as the *euclidean* metric. Proving that this function is indeed a metric is not altogether a trivial matter. Now, it should be clear that all metric properties but the triangle inequality hold for this function. Proving that the triangle inequality also holds requires some ingenuity. Those familiar with linear algebra have likely already seen the proof; it hinges on the well-known "dot product" for n-dimensional vectors. To see how, recall that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the formulas

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} x_j y_j \qquad \| \mathbf{x} \| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

define the dot product of two vectors and the *norm* (or magnitude)of a vector. It is a routine matter to prove that for any real numbers a and b, we have

$$\mathbf{x} \cdot (a\mathbf{y} + b\mathbf{z}) = a\mathbf{x} \cdot \mathbf{y} + b\mathbf{x} \cdot \mathbf{z} \qquad \| a\mathbf{x} \| = |a| \| \mathbf{x} \|$$

where, of course, $a\mathbf{x} = (ax_1, ..., ax_n)$. The previous properties give us the famous *Cauchy-Schwartz inequality*

$$|\mathbf{x} \cdot \mathbf{y}| \leq \| \mathbf{x} \| \| \mathbf{y} \|$$

To see why, first note that the inequality is trivially true if $\mathbf{y}$ is the zero-vector (all components are zero). If this vector is nonzero, then observe that

$$\| \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\mathbf{y} \| \geq 0 \implies (\mathbf{x} \cdot \mathbf{y})^2 \leq (\| \mathbf{x} \| \| \mathbf{y} \|)^2$$

The Cauchy-Schwartz inequality follows at once from the latter inequality. Now, by expanding $(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y})$ and applying the Cauchy-Schwartz inequality, we obtain

$$\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|$$

Because $E(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \|$, we may conclude that the function E does indeed satisfy the triangle inequality, since we have

$$E(\mathbf{x}, \mathbf{y}) = \| (\mathbf{x} - \mathbf{z}) + (\mathbf{z} - \mathbf{y}) \| \leq \| \mathbf{x} - \mathbf{z} \| + \| \mathbf{z} - \mathbf{y} \| = E(\mathbf{x}, \mathbf{z}) + E(\mathbf{z}, \mathbf{y})$$

Exercise 1.7.7. Let X be any nonempty set. Define a function $\mu : X \times X \longrightarrow \mathbb{R}$ by letting $\mu(x, x) = 0$ and $\mu(x, y) = \mu(y, x)$ be any number in the interval $[1, 2]$ when $x \neq y$. Prove that μ is a metric.

Exercise 1.7.8. Let $\mathcal{F}$ denote the set of all continuous functions on the closed interval $[0, 1]$ (relative to the usual subspace topology). Prove that the following are metrics on $\mathcal{F}$.

1. $\mu_1(f, g) = \mathtt{Max}\{|f(x) - g(x)| : x \in [0, 1]\}$
2. $\mu_2(f, g) = \displaystyle\int_0^1 |f(x) - g(x)|\, dx$

Definition 1.7.9. Let X be any set and let μ be a metric on X. For all $x \in X$ and all real $r > 0$, let $S_r(x) = \{(y \in X : \mu(x, y) < r\}$. We call $S_r(x)$ the *sphere of radius r centered on x* (relative to μ).

It might be interesting to note that in $\mathbb{R}^2$, spheres centered at a point (a, b) will be disks under the euclidean metric, the interior of diamonds under the taxicab metric, and the interior of squares under the M metric.

Theorem 1.7.10. *Let X be a set and let μ be a metric on X. The set $\mathcal{S} = \{\emptyset\} \cup \{S_r(x) : x \in X, r \in \mathbb{R}\}$ forms a basis for a topology on X. (We say that μ generates this topology.)*

Exercise 1.7.11. Prove Theorem 1.7.10.

Let X be a set and let μ be a metric on X. If we let Ω_μ denote the topology generated by μ, then the pair (X, Ω_μ) is called the *metric space induced by μ*.

Theorem 1.7.12. *Suppose that μ and ν are metrics on a set X. If there exists a constant k such that $\mu(a, b) \leq k\nu(a, b)$ for all $a, b \in X$, then $\Omega_\mu \subseteq \Omega_\nu$.*

Proof. Let $U \in \Omega_\mu$. To show that $U \in \Omega_\nu$, it will suffice to show that for each $x \in U$ there exist $V \in \Omega_\nu$ such that $x \in V \subseteq U$. To this end, let $x \in U$ and let r be such that $S_r(x) \in \Omega_\mu$ be such that $S_r(x) \subseteq U$. Let $t = r/k$ and consider the set $S_t(x) \in \Omega_\nu$. Clearly $x \in S_t(x)$. Now, if $y \in S_t(x)$, then we know $\nu(x, y) < r/k$. This tells us

$$\mu(x, y) \leq k\nu(x, y) < k\left(\frac{r}{k}\right) = r$$

Hence, we know that $y \in S_r(x)$; and we may conclude that $S_t(x) \subseteq S_r(x)$. This proves that $U \in \Omega_\nu$, as desired.

$\square$

Exercise 1.7.13. Show that the following inequalities hold in $\mathbb{R}^n$:

1. $M[\mathbf{x}, \mathbf{y}] \leq T[\mathbf{x}, \mathbf{y}] \leq nM[\mathbf{x}, \mathbf{y}]$
2. $M[\mathbf{x}, \mathbf{y}] \leq E[\mathbf{x}, \mathbf{y}] \leq \sqrt{n}M[\mathbf{x}, \mathbf{y}]$

Exercise 1.7.14. Use Theorem 1.7.12 to prove that the metrics E, M and T in Exercise 1.7.6 generate the same topology on $\mathbb{R}^n$.

Exercise 1.7.15. Prove that the metric space on $\mathbb{R}^n$ generated by the M metric is the usual product topology on $\mathbb{R}^n$.

Exercise 1.7.16. Let X be any nonempty set. Prove that the metric space generated by the discrete metric δ is the same as the discrete topology on X. (See Exercises 1.2.2 and 1.7.2.)

Let (X, Ω) be a topological space. Whenever there exists a metric μ on X such that $\Omega = \Omega_\mu$, we say that (X, Ω) is a *metrizable* space. Note that the discrete topology on any set is metrizable, as is the usual product topology on $\mathbb{R}^n$. We will have a lot more to say about metrizable spaces in the next section.

Theorem 1.7.17. *If $(X\Omega_\mu)$ and (Y, Ω_ν) are metric spaces, then the product space $(X \times Y, \prod)$ relative to Ω_μ and Ω_ν is metrizable.*

Exercise 1.7.18. Prove Theorem 1.7.17 Hint: Consider the metric $S[(a, b), (c, d)] = \mu(a, c) + \nu(b, d)$.

Exercise 1.7.19. Suppose that (X, Ω) and (Y, Θ) are homeomorphic topological spaces and suppose that $f : X \longrightarrow Y$ is a homeomorphism. Prove that if (X, Ω) is metrizable then (Y, Θ) is metrizable. Hint: Consider the mapping $\nu : Y \times Y \longrightarrow \mathbb{R}$ defined by $\nu(a, b) = \mu(f^{-1}(a), f^{-1}(b))$, where μ is a metric on X that generates Ω.

Exercise 1.7.20. Let (X, Ω_μ) be a metric space and let $\rho : X \times X \longrightarrow \mathbb{R}$ be defined by $\rho(a, b) = \mathtt{Min}(1, \mu(a, b))$.

1. Prove that ρ is a metric on X.
2. Prove that ρ is *bounded*; that is, there exists a real number M such that $\rho(a, b) \leq M$ for all $a, b \in X$.
3. Prove that $\Omega_\rho = \Omega_\mu$.

The previous exercise tells us that if μ is any metric on a set X, there is a bounded metric on X that generates the same topology. Most of the time, this fact is little more than a curiosity, but there are occasions when it is vital. We now consider an example.

In Exercise 1.6.29 we introduced the set $\mathbb{R}^\infty$ of all sequences of real numbers and considered the usual product and box topologies on this set. (In particular, we showed that these topologies are not equal.) Since the usual product (and box) topology on $\mathbb{R}^n$ is metrizable for each $n \in \mathbb{Z}^+$, it is reasonable to wonder whether this is true for $\mathbb{R}^\infty$ as well. Of course, since these topologies are distinct for $\mathbb{R}^\infty$, this question is more subtle. It is further complicated by the fact that the intuitive extension of the Euclidean metric to $\mathbb{R}^\infty$ does not make sense for all pairs of sequences, since we must deal with series convergence issues.

Exercise 1.7.21. Consider the bounded metric $\nu : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\nu(a, b) = \mathtt{Min}(1, |a - b|)$. This metric generates the usual topology on $\mathbb{R}$. Prove that

$$u(\mathbf{x}, \mathbf{y}) = \mathtt{Sup}\{\nu(x_n, y_n) : n \in \mathbb{Z}^+\}$$

defines a metric on $\mathbb{R}^\infty$. (This is called the *uniform* metric on this set.)

Recall that a basis for the product topology Ω_p on $\setminus^\infty$ is the family of all products of collections

$$\mathcal{U} = \{U_n : n \in \mathbb{Z}^+, U_n \in \Omega \text{ and all but finitely many } U_n = \mathbb{R}\}$$

while a basis for the box topology Ω_b is simply the family of all products of collections

$$\mathcal{U} = \{U_n : n \in \mathbb{Z}^+, U_n \in \Omega\}$$

The uniform metric on $\mathbb{R}^\infty$ generates the *uniform* topology on this set. It turns out that the uniform topology is neither the box topology nor the product topology on $\mathbb{R}^\infty$, but rather lies *between* them. In other words, if we let Ω_u denote the uniform topology on $\mathbb{R}^\infty$, then $\Omega_p \subset \Omega_u \subset \Omega_b$.

To see why this is so, first suppose that $U \in \Omega_p$ and suppose $\mathbf{x} \in U$. Let $j_1, ..., j_n$ be the indices for which $U_{j_i} \neq \mathbb{R}$. For each j_i, there exists a real number r_i such that the basic open $S_{r_i}(x_{j_i})$ (relative to the bounded metric ν) is contained in U_{j_i}. Let

$$r = \texttt{Min}\{r_1, ..., r_n\}$$

and consider the basic open $S_r(\mathbf{x})$ (relative to the uniform metric u). If $\mathbf{z} \in \mathbb{R}^\infty$ is such that $u(\mathbf{x}, \mathbf{z}) < r$, then by choice of r, we know that $\nu(x_{j_i}, y_{j_i}) < r_i$. We may therefore conclude that $S_r(\mathbf{x}) \subseteq U$. It follows that U can be written as the union of a family of basic opens from the uniform topology; consequently, $U \in \Omega_u$.

On the other hand, suppose that $V \in \Omega_u$ and suppose that $\mathbf{x} \in V$. There exists a real number r such that the basic open $S_r(\mathbf{x})$ relative to the uniform metric u) is contained in V. Consider the box product of segments

$$U = \prod\{(x_i - r, x_i + r) : i \in \mathbb{Z}^+\}$$

This set is clearly a member of Ω_b. Furthermore, if $\mathbf{y} \in U$, then $\nu(x_i, y_i) < r/2$ for all $i \in \mathbb{Z}^+$; hence we know that $u(\mathbf{x}, \mathbf{y}) \leq r/2$. This tells us that $\mathbf{y} \in S_r(\mathbf{x})$; and this tells us that $U \subseteq S_r(\mathbf{x})$. Consequently, we know that V can be written as the union of a family of basic opens from the box topology. We may conclude that $V \in \Omega_b$, as desired.

We have shown that $\Omega_p \subseteq \Omega_u \subseteq \Omega_b$. It remains to prove these inclusions are proper. To this end, for each $j \in \mathbf{Z}^+$ let $X_j = (-1/2, 1/2)$ and consider the set

$$(-1/2, 1/2)^\infty = \prod\{X_j : j \in \mathbb{Z}^+\}$$

The set $(-1/2, 1/2)^\infty$ is open in the box topology. Now, let $r \in (0, 1)$ and consider the sequences

$$\mathbf{x} = \left\{\frac{1}{2} - \frac{1}{j} : j \in \mathbb{Z}^+\right\} \qquad \mathbf{y} = \left\{\frac{1}{2} + \frac{rj - 2}{2j} : j \in \mathbb{Z}^+\right\}$$

First, note that there exists some positive integer n such that $nr - 2 > 0$. Consequently, we know that $\mathbf{y} \notin (-1/2, 1/2)^\infty$. Now, observe that

$$\left|\left(\frac{1}{2}-\frac{1}{j}\right)-\left(\frac{1}{2}+\frac{rj-2}{2j}\right)\right|=\frac{r}{2}$$

It follows that $\mathbf{y} \in S_r(\mathbf{x})$ (relative to the uniform metric). Consequently, any uniform basic open centered on $\mathbf{x}$) contains an element that is not in the set $(-1/2, 1/2)^\infty$; and we may conclude that $(-1/2, 1/2)^\infty$ is not open in the uniform topology. This proves that $\Omega_u \subset \Omega_b$.

Exercise 1.7.22. Explain why any basic open in the product topology on $\mathbb{R}^\infty$ must contain sequences $\mathbf{x}$ such that $u(\mathbf{0}, \mathbf{x}) = 1$. Use this observation to explain why the uniform basic open $S_{1/2}(\mathbf{0})$ is not open in the product topology on $\mathbb{R}^\infty$. (Hence, we know that $\Omega_p \subset \Omega_u$.)

We now have three distinct topologies on $\mathbb{R}^\infty$, one of which is a metric space. It turns out that the product topology is also metrizable, although the metric which accomplishes this are more subtle. The box topology on $\mathbb{R}^\infty$ is *not* metrizable; however, we cannot yet prove this fact.

Exercise 1.7.23. Prove that the function $p : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ defined below is a metric.

$$p(\mathbf{x}, \mathbf{y}) = \mathtt{Sup}\left\{\frac{\nu(x_j, y_j)}{j} : j \in \mathbb{Z}^+\right\}$$

Exercise 1.7.24. Consider the set $\mathbb{R}^\infty$, under the metric p. For each $\mathbf{x} \in \mathbb{R}^\infty$, let $S_r(\mathbf{x})$ be a basic open under p. Choose $N \in \mathbb{Z}^+$ such that $N^{-1} < r$. Show that the basic product open $V = \prod\{X_j : j \in \mathbb{Z}^+$ is contained in $S_r(\mathbf{x})$, where $X_j = (x_j - r, x_j + r)$ for $1 \le j \le N$, and $X_j = \mathbb{R}$ otherwise.

Exercise 1.7.25. Consider the set $\mathbb{R}^\infty$ under the product topology. For each $\mathbf{x} \in \mathbb{R}^\infty$, let $U = \prod\{X_j : j \in \mathbb{Z}^+$ be a basic product open containing $\mathbf{x}$. For each index j such that $X_j \neq \mathbb{R}$, let r_j be such that the segment $(x - r_j, x + r_j) \subseteq X_j$. Let r be the smallest of the quotients r_j/j and prove that the basic open $S_r(\mathbf{x})$ under the p metric is contained in U.

Exercises 1.7.24 and 1.7.25 together tell us that the product topology on $\mathbb{R}^\infty$ is the same as the topology generated by the metric p. Hence, this product topology is metrizable. If J is an uncountable index set, then the uniform metric u can easily be extended to the set $\mathbb{R}^J$; however the product topology on this set is *not* metrizable. Machinery needed to establish this fact will be developed in the next section.

We conclude this section with a result that gives us an easy way to construct new metrics from existing ones. Let $X \subseteq \mathbb{R}$. A function $f : X \longrightarrow \mathbb{R}$ is *concave* provided for any $a, b \in X$, we have

$$f(ta + (1-t)b) \geq tf(a) + (1-t)f(b)$$

for any $t \in [0,1]$. This condition merely says that the graph of f lies above the straight line between the points $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem from calculus can be used to prove that any twice-differentiable function with negative second derivative is concave, so these functions are quite common.

Exercise 1.7.26. Suppose that $f : [0, +\infty) \longrightarrow \mathbb{R}$ is concave and suppose that $f(0) \geq 0$.

1. Prove that for $t \in [0,1]$ and $x \in [0, +\infty)$ we have $f(tx) \geq tf(x)$.
2. For $a, b \in [0, +\infty)$, prove that $f(a+b) \leq f(a) + f(b)$. (Hint: Observe that $a = (a+b)\dfrac{a}{a+b}$ and $b = (a+b)\dfrac{b}{a+b}$.)

Functions with this property are said to be *subadditive*.

Exercise 1.7.27. Let (X, μ) be any metric space and suppose that $f : [0, +\infty) \longrightarrow \mathbb{R}$ is subadditive. If $f(0) = 0$, $f(x) > 0$ for $x > 0$, and $x \leq y$ implies $f(x) \leq f(y)$, then prove that $f \circ \mu$ is also a metric on X.

Exercise 1.7.28. Let (X, μ) be any metric space. Use the previous exercises to prove that the following are also metrics on X.

$$L(x,y) = \sqrt{\mu(x,y)} \qquad M(x,y) = \frac{\mu(x,y)}{1+\mu(x,y)} \qquad N(x,y) = \ln(1+\mu(x,y))$$

1.8 Properties of Metric Spaces

Having introduced some examples of metric spaces, we will now explore properties of metric spaces. In particular, we will examine the separation properties that metric spaces possess, what conditions lead to first and second countability and to separability and will provide a powerful tool for determining when certain topological spaces are metrizable. We begin by noting that the real numbers under the discrete topology (which is a metric space under the discrete metric δ by Exercise 1.7.16) is neither second countable nor separable.

Theorem 1.8.1. *Every compact metric space (X, Ω_μ) is separable.*

Proof. For each $n \in \mathbb{Z}^+$, let $r = n^{-2}$. The family $\mathcal{F} = \{S_r(x) : x \in X\}$ is clearly an open cover for X. Since X is compact, we know each such cover admits a finite subcover $\mathcal{F}_n = \{S_r(x_i) : 1 \leq i \leq n_r\}$. For each n, let $C_n = \{x_1^{(n)}, ..., x_{n_r}^{(n)}\}$. Observe that $\mu(a,b) > n^{-2}$ for all $a, b \in C_n$. Consider the set $D = \bigcup\{C_n : n \in \mathbb{Z}^+\}$. This set is clearly countable. We will prove that it is also dense in X. To this end, let $y \in X - D$ and suppose that $U \in \Omega_\mu$ contains y. There exists $m \in \mathbb{Z}^+$ such that $S_{1/m}(y) \subseteq U$. Let $r = (2m)^{-2}$. Now, $\mathcal{F}_{2m}$ is a cover for X, so we know there exist $x_j^{(2m)}$ such that $y \in S_r(x_j^{(2m)})$. Consequently, we know that $\mu(y, x_j^{(2m)} \leq (2m)^{-2} < m^{-1}$. Therefore, we know that $D \cap U \neq \emptyset$. It follows that y must be a limit point of D, and this tells us that $X = \overline{D}$.

□

Theorem 1.8.2. *Every metric space is first countable, and every separable metric space is second-countable.*

Exercise 1.8.3. Prove Theorem 1.8.2.

We should note that the Sorgenfrey line is not a metrizable space, since it is separable (Exercise 1.5.10) but not second countable (Theorem 1.5.34). Consequently, there do exist topological spaces which are not metrizable.

Exercise 1.8.4. Prove that every singleton subset of a metric space is closed. (Hence every metric space is $T1$ by Theorem 1.4.5.)

Exercise 1.8.5. Prove that every metric space is Hausdorff.

Definition 1.8.6. A topological space (X, Ω) has the T_4 separation property if, for each pair of disjoint $H, K \in \kappa(\Omega)$, there exist disjoint $U, V \in \Omega$ such that $H \subseteq U$ and $K \subseteq V$. A topological space that is both T_1 and T_4 is said to be *normal*.

Exercise 1.8.7. Let (X, Ω) be a normal topological space. If H and K are disjoint members of $\kappa(\Omega)$, then prove there exist $U, V \in \Omega$ such that $H \subseteq U$, $K \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Theorem 1.8.8. *If (X, Ω) is a T_1 topological space, then the following statements are equivalent:*

1. *The space (X, Ω) is normal.*
2. *Each pair of disjoint members of $\kappa(\Omega)$ have disjoint neighborhoods.*
3. *For each $H \in \kappa(\Omega)$ and each $U \in \Omega$ such that $H \subseteq U$, there exist $V \in \Omega$ such that $H \subseteq V$ and $\overline{V} \subseteq U$.*
4. *If H and K are disjoint members of $\kappa(\Omega)$, there exist $U \in \Omega$ such that $H \subseteq U$ and $\overline{U} \cap K = \emptyset$.*

Exercise 1.8.9. Prove Theorem 1.8.8.

Definition 1.8.10. Let (X, Ω) be a topological space. Two subsets H and K of X are *separated* provided $H \cap \overline{K} = K \cap \overline{H} = \emptyset$.

Definition 1.8.11. Let (X, Ω) be a topological space. We say that (X, Ω) is *completely normal* provided (X, Ω) is T_1 and, whenever H and K are separated subsets of X, there exist disjoint $U, V \in \Omega$ such that $H \subseteq U$ and $K \subseteq V$.

Exercise 1.8.12. Prove that every metric space is completely normal.

Definition 1.8.13. Let (X, Ω_μ) be a metric space. A subset A of X is *bounded* relative to μ provided there exist $k > 0$ such that $\mu(x, y) \leq k$ for all $x, y \in A$. The number k is called a *bound* for A.

Definition 1.8.14. Let (X, Ω_μ) be a metric space. The *diameter* of a bounded subset A of X is the smallest bound for A. We let $\Delta(A)$ denote the diameter of A. If A is not bounded, we say that A has infinite diameter.

Let (X, Ω_μ) be a metric space, and let A and B be nonempty subsets of X and let

$$D(A, B) = \{\mu(a, b) : a \in A, b \in B\}$$

Since μ is a metric, we know that $0 \leq \mu(a, b)$ for all $a \in A$ and $b \in B$. It follows that the set $D(A, B)$ has a greatest lower bound. We define $\mu(A, B)$ to be this number. If one of A or B is empty, we let $\mu(A, B) = 0$.

Exercise 1.8.15. Let (X, Ω_μ) be a metric space. If A and B are subsets of X, then prove that the following statements are true:

1. We have $0 \leq \mu(A, B) \leq \mu(a, b) \leq \Delta(A \cup B)$ for all $a \in A$ and $b \in B$.
2. We have $\Delta(A \cup B) \leq \Delta(A) + \Delta(B) + \mu(A, B)$.
3. We have $\mu(\{a\}, B) = 0$ if and only if $B = \emptyset$ or $a \in \overline{B}$.
4. If A and B are nonempty, compact, and disjoint, then $0 < \mu(A, B)$. Moreover, there exist $a \in A$ and $b \in B$ such that $\mu(a, b) = \mu(A, B)$.

Exercise 1.8.16. Find nonempty, closed, disjoint subsets A and B of $\mathbb{R} \times \mathbb{R}$ (under the usual topology) such that $\mu(A, B) = 0$.

Theorem 1.8.17. *Let (X, Ω_μ) be a metric space. If A is a compact subset of X, then the following statements are true:*

1. *The set A is bounded.*
2. *The set A is closed.*
3. *If A is nonempty, then there exist $x, y \in A$ such that $\mu(x, y) = \Delta(A)$.*

Proof. Suppose A is compact. The set $\mathcal{F} = \{S_1(a) : a \in A\}$ is clearly an open cover for A. Let $a_1, ..., a_n$ denote elements in A such that $\mathcal{F}' = \{S_1(a_1), ..., S_1(a_n)\}$ covers A. Let $x, y \in A$. It follows that $x \in S_1(a_j)$ and $y \in S_1(a_k)$ for some $1 \leq j, k \leq n$. Now, observe that

$$\mu(x, y) \leq \mu(x, a_j) + \mu(a_j, y) \leq \mu(x, a_j) + \mu(a_j, a_k) + \mu(a_k, y) < 2 + \mu(a_j, a_k)$$

If we let $n = \mathtt{max}\{\mu(a_l, a_m) : 1 \leq l, m \leq n\}$, then it is clear that $\mu(x, y) \leq 2 + n$. Thus, we know that there is an upper bound for $\mu(x, y)$ for all $x, y \in A$. Hence, A is bounded.

We now prove that the set A is closed. We know that every metric space is Hausdorff; hence, we know that A is closed by Exercise 1.4.17.

Let $\Delta(A)$ denote the diameter for A. If A is finite, then there is nothing to show, so suppose that A is infinite. Let $x_1, y_1 \in A$. If $\mu(x_1, y_1) = \Delta(A)$, there is nothing to show, so suppose that $\mu(x_1, y_1) < \Delta(A)$. Since $\Delta(A)$ is by definition the *least* of the upper bounds for A, there must $x_2, y_2 \in A$ such that

1. $\mu(x_1, y_1) < \mu(x_2, y_2)$ and
2. $\Delta(A) - (1/2)\Delta(A) < \mu(x_2, y_2)$.

We can continue in like fashion, creating the sequences $x, y \subseteq A$ such that $\mu(x_n, y_n) < \mu(x_{n+1}, y_{n+1})$ and $\Delta(A) - (1/2^{n-1})\Delta(A) < \mu(x_n, y_n)$. If it is ever the case that $\mu(x_{n+1}, y_{n+1}) = \Delta(A)$, the process stops; and we are done. If this is not the case, the both sequences x and y are infinite.

Now, since A is a compact subset of a second-countable space, we know that both sequences contain convergent subsequences by Theorem 1.5.36. Let x_0 and y_0 be the limits, respectively, for the sequences x and y. Since A is closed, we know that $x_0, y_0 \in A$. Since μ is continuous, we know that the sequence $\mu(x, y)$ converges to $\mu(x_0, y_0)$. Since $\mu(x, y)$ converges to $\Delta(A)$ by construction, we are done.

□

Exercise 1.8.18. Let (X, Ω_μ) be a metric space. Prove that $\mu : X \times X \longrightarrow \mathbb{R}$ is continuous relative to the product topology on $X \times X$ and the usual topology on $\mathbb{R}$.

Exercise 1.8.19. Let (X, Ω_μ) and (Y, Ω_ρ) be metric spaces and suppose that $f : X \longrightarrow Y$ is a function. Prove that f is continuous if and only if, for every $x \in X$ and $\epsilon > 0$, there exist $\delta > 0$ such that $f(S_\mu(x)) \subseteq S_\epsilon(f(x))$.

Exercise 1.8.20. Let X be any nonempty set equipped with a function $D : X \times X \longrightarrow \mathbb{R}$ satisfying

1. $D(x, x) = 0$
2. $D(x, y) \neq 0$ if $x \neq y$
3. $D(x, y) \leq D(y, z) + D(z, x)$ for all $x, y, z \in X$

Prove that D is a metric.

Exercise 1.8.21. Let X be any nonempty set equipped with a metric μ. We say that μ is an *ultrametric* provided $\mu(x, z) \leq \mathtt{max}[\mu(x, y), \mu(y, z)]$ for all $x, y, z \in X$. Prove that the metric D in Exercise 1.7.7 is an ultrametric.

1.9 Urysohn's Lemma

Determining whether or not a topological space is metrizable is not easy in general. We conclude our discussion of metric spaces by developing a tool which enables us to determine when certain topological spaces are metrizable. In particular, we will prove *Urysohn's Lemma*, which is one of the deepest results in elementary topology. We begin with some intermediate results.

Definition 1.9.1. A *dyadic* rational number has the form $m/2^n$, where m and n are nonnegative integers. We will let

$$\mathbb{D} = \{\frac{m}{2^n} : m, n \in \mathbb{N}, m < 2^n\}$$

Note that $\mathbb{D}$ is the set of dyadic rationals in the half-open segment $[0, 1)$. Note also that 1 (which is a dyadic rational) is a limit point of $\mathbb{D}$ under the usual topology, since we have

$$\frac{2^n - 1}{2^n} \in \mathbb{D}$$

for all natural numbers n.

Theorem 1.9.2. *Let (X, Ω) be a normal topological space and let A and B be disjoint closed subsets of X. There exists a sequence $\mathcal{U} = \{U_t : t \in \mathbb{D}\} \subseteq \Omega$ such that, whenever $s < t$, then*

$$A \subseteq U_s \subseteq \overline{U_s} \subseteq U_t \subseteq \overline{U_t} \subseteq X - B$$

Proof. First, note that $X - B$ is open and that $A \subseteq X - B$. Since (X, Ω) is normal, we know there exists a set $U_{1/2} \in \Omega$ such that $A \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq X - B$ by Theorem 1.8.8 (3). We will let $\mathcal{S}_1 = \{U_{1/2}\}$ and call this the "first level" of the construction.

Now, let n be any natural number and let $0 < j < 2^n$. Let $a_j = j/2^n$ and suppose that the "nth level" of the construction has been made. That is, suppose that the family of opens $\mathcal{U}_n = \{U_{a_j} : 0 < j < n\}$ has been chosen such that, for each $0 < k < j$, we have

$$A \subseteq U_{a_k} \subseteq \overline{U_{a_k}} \subseteq U_{a_j} \subseteq \overline{U_{a_j}} \subseteq X - B$$

We will use induction to establish that the "$n + 1$st level" can be constructed. First, let $b_k = k/2^{n+1}$ for $0 < k < 2n + 1$. Now, either k is even or it is not. Suppose that k is even. It follows that $k = 2j$ for some $0 < j < 2^n$; hence, we know that

$$b_k = \frac{2j}{2^{n+1}} = \frac{j}{2^n}$$

Thus, if k is even, we will let $U_{b_k} = U_{a_j}$. Suppose instead that k is odd. Of course, this means that $k = 2j + 1$ for some $0 \leq j \leq 2^n - 1$. By Theorem 1.8.8 (3), we know we can select an open U_{b_1} such that

$$A \subseteq U_{b_1} \subseteq \overline{U_{b_1}} \subseteq U_{a_1}$$

Similarly, for each $0 < j < 2^n - 1$, we can select an open $U_{b_{2j+1}}$ such that

$$\overline{U_{a_j}} \subseteq U_{b_{2j+1}} \subseteq \overline{U_{b_{2j+1}}} \subseteq U_{a_{j+1}}$$

Finally, when $j = 2^n - 1$, we can select an open $U_{b_{2j+1}}$ such that

$$\overline{U_{a_j}} \subseteq U_{b_{2j+1}} \subseteq \overline{U_{b_{2j+1}}} \subseteq X - B$$

We have now constructed the set $\mathcal{U}_{n+1}$ from the set $\mathcal{U}_n$. It follows by induction that we may construct the set $\mathcal{U}_n$ for every natural number n. The desired set $\mathcal{U}$ is the union of these families.

□

Exercise 1.9.3. Let (X, Ω) be a normal topological space and let A and B be disjoint closed subsets of X. Let $\mathcal{U}$ be the sequence of opens constructed in the proof of Theorem 1.9.2. Define a map $f : X \longrightarrow [0, 1]$ by letting

$$f(x) = \begin{cases} 0 & \text{if} x \in U_t \text{ for all } t \in \mathbb{D}, \\ \mathtt{sup}\{t \in \mathbb{D} : x \notin U_t\} & \text{otherwise.} \end{cases}$$

Prove that f is indeed a function. Furthermore, prove that $x \in \overline{U_t}$ implies that $f(x) \leq t$ while $x \notin U_t$ implies that $t \leq f(x)$.

Exercise 1.9.4. Let (X, Ω) be a normal topological space and let A and B be disjoint closed subsets of X. Let $f : X \longrightarrow [0, 1]$ be as defined in Exercise 1.9.3. Show that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Exercise 1.9.5. Prove that every subspace of a metric space is itself a metric space.

Theorem 1.9.6. *(Urysohn's Lemma) A T_1 topological space (X, Ω) is normal if and only if, for each pair of disjoint $A, B \in \kappa(\Omega)$, there exists a function $f : X \longrightarrow [0, 1]$ such that $f(a) = 0$ for all $a \in A$, $f(b) = 1$ for all $b \in B$ and is continuous relative to Ω and the usual subspace topology on $[0, 1]$.*

Proof. First, suppose that such a function f exists for each pair of closed disjoint sets A and B. We know that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$. Note that since $U = [0, 1/2) = [0, 1] \cap (-1, 1)$, we know $[0, 1/2)$ is open in $[0, 1]$ under the subspace topology. Likewise, the set $V = (1/2, 1]$ is open in the subspace topology. These sets are clearly disjoint; since f is a function, we also know that $f^{-1}(U)$ and $f^{-1}(V)$ are also disjoint. Since f is continuous, we know that $f^{-1}(U)$ and $f^{-1}(V)$ are open in Ω. Since $A \subseteq f^{-1}(U)$ and $B \subseteq f^{-1}(V)$, we see that (X, Ω) has the T_4 separation property.

Conversely, suppose that (X, Ω) is normal. Let A and B be disjoint closed subsets of X and let $f : X \longrightarrow [0, 1]$ be defined as in Exercise 1.9.3. We only need to prove that f is continuous. Since basic opens in $\mathbb{R}$ under the usual topology are open segments, we know a subset U of $[0, 1]$ will be a basic open in the usual subspace topology if and only if it has the form $[0, b)$, (a, b), or $(a, 1]$ where $a, b \in [0, 1]$. Since each of these subsets can be represented as finite intersections of opens $[0, b)$ or $(a, 1]$, it will suffice to consider only sets of this type.

First, consider $U = [0, b)$. Let $\mathbb{D}$ denote the dyadic rationals. We will show that $f^{-1}(U) = \bigcup\{U_t : t < b\}$, where the sets U_t are constructed in the proof of Theorem 1.9.2. The set of dyadic rationals is dense in $[0, 1]$; hence, for each $x \in f^{-1}(U)$, we know that there exist $t \in \mathbb{D}$ such that $f(x) < t < b$. Recall that

$$f(x) = \begin{cases} 0 & \text{if} x \in U_t \text{ for all } t \in \mathbb{D}, \\ \mathtt{sup}\{t \in \mathbb{D} : x \notin U_t\} & \text{otherwise.} \end{cases}$$

Consequently, since $f(x)$ is the supremum of all the dyadic rationals s such that $x \notin U_s$, the fact that $f(x) < t$ implies $x \in U_t$. Hence, $f^{-1}(U) \subseteq \bigcup\{U_t : t < b\}$. For the reverse inclusion, suppose that t is a dyadic rational such that $t < b$. If $s \in \mathbb{D}$ is such that $t \leq s$, then we know that $U_t \subseteq U_s$. Thus, if $x \in U_t$, we know that $x \in U_s$. It follows that if $x \notin U_p$ for some p, then we must have $p < t$. Hence, $f(x) \leq t < b$; and it follows that $x \in f^{-1}(U)$.

Finally, consider $V = (a, 1]$. We will prove that $f{-}1(V) = \bigcup\{X - \overline{U_t} : a < t\}$. Suppose that $x \in f^{-1}(V)$ and let $s, t \in \mathbb{D}$ be such that $a < s \leq t < f(x)$. Now,

$U_s \subseteq \overline{U_t}$ by selection; hence. Also, we know that $x \notin U_s$ since $s < f(x)$. Hence, we know that $x \in X - \overline{U_t}$. It follows that $f^{-1}(U) \subseteq \bigcup\{X - \overline{U_t} : a < t\}$. For the reverse inclusion, suppose that $x \in \bigcup\{X - \overline{U_t} : a < t\}$. There exist $t \in \mathbb{D}$ such that $t > b$ and $x \notin \overline{U_t}$. It follows that $f(x) \geq t > b$. Hence, $x \in f^{-1}(U)$, as desired.

□

Exercise 1.9.7. Let (X, Ω) be a regular topological space (see Exercise 1.4.8). If (X, Ω) is second-countable, prove that (X, Ω) is normal.

Theorem 1.9.8. *Let (X, Ω) be a regular topological space which is second countable. There exists a countable collection of continuous functions $f_n : X \longrightarrow [0, 1]$ having the property that, for any $a \in X$ and any open U containing a, there exist n_a such that f_{n_a} is positive on U and is identically 0 on $X - U$.*

Proof. Let $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$ be a countable basis. Let $a \in X$ and let U be any open containing a. It follows that there exists a $B_m \in \mathcal{B}$ such that $a \in B_m \subseteq U$. Since (X, Ω) is regular, we can find $B_n \in \mathcal{B}$ such that $a \in \overline{B_n} \subset B_m$. (See Exercise 1.4.9.) Consider the set

$$S = \{(m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : \overline{B_n} \subset B_m\}$$

This set is countable. Now, by Urysohn's Lemma, for each $(m, n) \in S$, there exists a continuous function $g_{(m,n)} : X \longrightarrow [0, 1]$ such that $g_{(m,n)}(x) = 1$ for all $x \in \overline{B_n}$ and $g_{(m,n)}(x) = 0$ for all $x \in X - \overline{B_m}$. This countable collection of functions can be reindexed to serve as the desired functions f_n.

□

Exercise 1.9.9. Consider the topological space $(\mathbb{R}^\infty, \Omega_p)$. Let (X, Ω) be a second countable regular topological space and define a mapping $F : X \longrightarrow \mathbb{R}^\infty$ be defined by $\pi_n(F(x)) = f_n(x)$, where the functions f_n satisfy the conclusion of Theorem 1.9.8.

1. Show that F is an injection.
2. Show that F is continuous relative to Ω and the product subspace topology on $F(X)$.
3. Show that F^{-1} is continuous relative to the product subspace topology on $F(X)$ and Ω.

Exercise 1.9.9 proves that every regular second countable topological space is homeomorphic to a subspace of $(\mathbb{R}^\infty, \Omega_p)$ (in particular, the subspace $[0, 1]^\infty$). It follows from Exercise 1.7.19 that every regular second countable topological space is metrizable. This result is known as the *Urysohn Metrization Theorem.*

Exercise 1.9.10. Prove that a compact Hausdorff space is metrizable if and only if it is second countable.

Let I be an uncountable index set. For each $i \in I$, let $X_i = [0,1]$ endowed with the usual subspace topology and consider the set

$$[0,1]^I = \prod\{X_i : i \in I\}$$

endowed with the product topology. This topological space is compact by Tychonoff's Theorem and is Hausdorff by Exercise 1.6.34. Exercise 1.6.35 tells us that $[0,1]^I$ is not first countable, and therefore not second countable, either. (Note — Exercise 1.6.35 requires the spaces to be *disjoint* but Exercise 1.6.11 provides a way around this.) We may therefore conclude that $[0,1]^I$ is not metrizable under the product topology.

Exercise 1.9.11. A topological space is *locally metrizable* provided each point is contained in an open set that is metrizable as a subspace. Prove that a locally metrizable compact Hausdorff space is metrizable.

Let $\mathcal{H}$ denote the set of all sequences $\mathbf{s}$ of real numbers such that $\sum_j^\infty s_j^2 < +\infty$. (These are sometimes called *square-summable* sequences.) Define a mapping $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ by

$$\eta[\mathbf{s}, \mathbf{t}] = \sqrt{\sum_{j=1}^{\infty} |s_j - t_j|^2}$$

The properties of convergent series developed in calculus can be used to prove that η is a metric on $\mathcal{H}$, although we will not do so here. The topological space $(\mathcal{H}, \Omega_\eta)$ is called the *Hilbert Space* or the ℓ^2 space. It turns out that on $\mathcal{H}$, we have $\Omega_p \subset \Omega_\eta \subset \Omega_u$.

Definition 1.9.12. Let $\mathcal{Q}$ denote the set of all sequences $\mathbf{s}$ of real numbers such that $|s_j| \leq 1/j$ for all $j \in \mathbb{Z}^+$. This particular set is a subspace of $\mathcal{H}$ under the metric η; we call it the *Hilbert Cube*.

Exercise 1.9.13. Prove that $([-1,1]^\infty, \Omega_p)$ is homeomorphic to the Hilbert Cube.

We conclude this section with another powerful application of Urysohn's Lemma. Before we present this result, however, we need a few ideas from analysis.

Definition 1.9.14. Let X be any set, and let (Y, Θ) be a metric space with metric μ. Let $F = \{f_n : n \in \mathbb{Z}^+\}$ be a sequence of functions $f_n : X \longrightarrow Y$, and let $f : X \longrightarrow Y$. We say that F *converges uniformly* to the function f provided for any $\epsilon > 0$, there exists an integer N such that $\mu(f_n(x), f(x)) < \epsilon$ for all $n > N$ and all $x \in X$.

Theorem 1.9.15. *Let (X, Ω) be a topological space and let (Y, Θ) be a metric space. Suppose that $F = \{f_n : n \in \mathbb{Z}^+\}$ is a sequence of continuous functions $f_n : X \longrightarrow Y$, and suppose $f : X \longrightarrow Y$. If F converges to f uniformly, then f is also continuous.*

Proof. Let $V \in \Theta$ and suppose that $a \in f^{-1}(V)$. It will suffice to find $U \in \Omega$ such that $a \in U$ and $f(U) \subseteq V$. (See Theorem 1.3.11.) Let $b = f(a)$ and select $\epsilon > 0$ such that the ball $S_\epsilon(b) \subseteq V$. Since F converges uniformly to f, we can select $N \in \mathbb{Z}^+$ such that $n > N$ guarantees $\mu(f(x), f_n(x)) < \epsilon/3$ for all $x \in X$. Of course, this implies that $\mu(f(b), f_n(b)) < \epsilon/3$ as well. Furthermore, since the function f_N is continuous, there is a $U \in \Omega$ containing a such that $f_N(U) \subseteq S_{\epsilon/3}(f_N(a))$. The triangle inequality (applied twice) for the metric μ now tells us that, for all $x \in U$,

$$\mu(f(x), f(a)) \leq \mu(f(x), f_N(x)) + \mu(f_N(x), f_N(a)) + \mu(f_N(a), f(a)) < \epsilon$$

Hence, we may conclude that $f(U) \subseteq S_\epsilon(a) \subseteq V$, as desired.

□

The previous result is known as the *uniform extension theorem* and is an important result in analysis. For our purposes, it plays a key role in proving *Tietze's Extension Theorem*, another consequence of Urysohn's Lemma. Here is the statement of the theorem.

Theorem 1.9.16. *Let $[a, b] \subset \mathbb{R}$ be endowed with the usual subspace topology and suppose that (X, Ω) is a normal topological space. If A is a closed subspace of X and $f : A \longrightarrow [a, b]$ is continuous, then there is a continuous function $g : X \longrightarrow [a, b]$ such that $g(x) = f(x)$ for all $x \in A$.*

The function g is called an *extension* or a *continuation* of the function f. We will develop the proof in the following exercises. In light of Exercise 1.6.13, we can replace the arbitrary closed interval $[a, b]$ with the interval $[-1, 1]$. The general idea will be to construct a sequence F of continuous functions from X to $[-1, 1]$ which converge uniformly to some function g such that each successive member of the sequence forms a better approximation to f on A. All of the following exercises assume that (X, Ω) is a normal topological space, A is a closed subset of X, and any interval $[a, b]$ is endowed with the usual subspace topology.

Exercise 1.9.17. Let $r > 0$ and suppose that $f : A \longrightarrow [-r, r]$ is continuous. Let

$$I_1 = \left[-r, -\frac{1}{3}r\right] \quad I_2 = \left[-\frac{1}{3}r, \frac{1}{3}r\right] \quad I_3 = \left[\frac{1}{3}r, r\right]$$

Let $B = f^{-1}(I_1)$ and $C = f^{-1}(I_3)$.

1. Explain why B and C are disjoint, closed subsets of A.

2. Invoke Urysohn's Lemma to show there must exist a continuous function $\varphi : X \longrightarrow I_2$ such that $\varphi(x) = -r/3$ for all $x \in B$ and $\varphi(x) = r/3$ for all $x \in C$.

3. For each $x \in A$, show that $|\varphi(x) - f(x)| \leq 2r/3$. (Consider the cases $x \in B$, $x \in C$, $x \in A - (B \cup C)$ separately.)

Suppose now that $f : X \longrightarrow [-1, 1]$ is continuous. The previous exercise tells us that there exists a continuous function $g_1 : X \longrightarrow [-1, 1]$ such that

$$|g_1(x)| \leq \frac{1}{3} \quad (x \in X) \qquad |f(x) - g_1(x)| \leq \frac{2}{3} \quad (x \in A)$$

Now, $f - g_1 : X \longrightarrow [-2/3, 2/3]$. Consequently, there exists a continuous function $g_2 : X \longrightarrow [-1, 1]$ such that

$$|g_2(x)| \leq \frac{2}{9} \quad (x \in X) \qquad |f(x) - g_1(x) - g_2(x)| \leq \frac{4}{9} \quad (x \in A)$$

This process may be repeated indefinitely to create a sequence of continuous functions g_n such that

$$|g_n(x)| \leq \frac{2^{n-1}}{3^n} \quad (x \in X) \qquad |f(x) - g_1(x) - ... - g_n(x)| \leq \left(\frac{2}{3}\right)^n \quad (x \in A)$$

Exercise 1.9.18. Let $g : X \longrightarrow \mathbb{R}$ be defined by

$$g(x) = \sum_{j=1}^{\infty} g_j(x)$$

1. Use an appropriate convergence test from calculus to prove that the function g is well-defined.

2. Show that g maps X to the interval $[-1, 1]$.

3. Let $n \in \mathbb{Z}^+$ and suppose $k > n$. Use an appropriate geometric series to show that
$$\left| \sum_{j=n+1}^{k} g_j(x) \right| < \left(\frac{2}{3}\right)^n$$

Exercise 1.9.19. Use the previous exercise to prove that the sequence of partial sums from the series definition of g converges uniformly, so that the function g is continuous.

Exercise 1.9.20. Use partial sums and what we know about $f - (g_1 + ... + g_n)$ to establish that $f(x) = g(x)$ for all $x \in A$.

Tietze's Extension Theorem also applies to continuous functions that map closed subsets of normal spaces to $\mathbb{R}$ under the usual topology. In other words, if (X, Ω) is a normal space, $A \subseteq X$ is closed, and $f : A \longrightarrow \mathbf{R}$ is continuous, then there is a continuous extension $g : X \longrightarrow \mathbb{R}$ for f.

Most of the work in proving this result has already been done. To begin, recall that the open segment $(-1, 1)$ under the usual subspace topology is homeomorphic to $\mathbb{R}$ under the usual topology. (See Exercises 1.6.14 and 1.6.15.) Consequently, we may assume that $f : A \longrightarrow (-1, 1)$; and it will suffice to extend f to a continuous function $g : X \longrightarrow (-1, 1)$. Of course, we know $f : A \longrightarrow [-1, 1]$; so we already know there exists a continuous extension $h : X \longrightarrow [-1, 1]$ for f. Let

$$D = h^{-1}(-1) \cup h^{-1}(1)$$

If $D = \emptyset$, then we can let $g = h$ and are done. Regardless, we know D is a closed subset of A since h is continuous; and since $h(A) = f(A)$, we know $A \cap D = \emptyset$. By Urysohn's Lemma, we know there exists a continuous function $\varphi : X \longrightarrow [0, 1]$ such that $\varphi(x) = 0$ for all $x \in D$ and $\varphi(x) = 1$ for all $x \in A$. The function $g = \varphi \cdot h$ is the extension that we seek.

1.10 Connectedness

A topological space (X, Ω) is said to be *connected* provided it is impossible to express X as the union of two disjoint nonempty open subsets. (We say the space is *disconnected* or *separated* otherwise.) Whenever such subsets do exist, we say they constitute a *separation* of X relative to Ω. The simple notion of connectedness turns out to have useful consequences — the Intermediate Value Theorem (Theorem 1.3.9) actually relies on the fact that the real numbers for a connected space under the usual topology.

Exercise 1.10.1. Suppose that (X, Ω) is a disconnected space and suppose $\{H, K\}$ is a *disconnection* of X. (That is, H and K are disjoint open sets such that $X = H \cup K$.) Prove that H and K are also closed.

Exercise 1.10.2. Let (X, Ω) be a topological space. Prove that this space is disconnected if and only if there exist nonempty closed sets A and B such that $A \cap B = \emptyset$ and $A \cup B = X$.

Theorem 1.10.3. *Consider the real numbers under the usual topology, and let $X \subseteq \mathbb{R}$ be endowed with the subspace topology. The following statements are equivalent:*

1. *The set X is connected as a subspace of $\mathbb{R}$.*
2. *Whenever $a, b \in X$ and $c \in \mathbb{R}$ is such that $a < c < b$, then $c \in X$.*

Proof. To prove that Claim (1) implies Claim (2), we will work with the contrapositive. In particular, suppose that X does not satisfy Claim (2). We will prove that X is disconnected. To this end, observe that there must exist $a, b \in X$ and $c \in \mathbb{R} - X$ such that $a < c < b$. Note that $H = X \cap (-\infty, c)$ and $K = X \cap (c, +\infty)$ are disjoint open sets in the subspace topology such that $X \subseteq H \cup K$. Hence, we may conclude that X is disconnected.

To prove that Claim (2) implies Claim (1), suppose by way of contradiction that X satisfies Claim (2), but a disconnection $\{H, K\}$ of X exists. Both H and K are nonempty, so let $a \in H$ and $b \in K$. Without loss of generality, we can assume $a < b$. Now, let $S = \{x \in H : x < b\}$. Since b is an upper bound for S, we know by Exercise 1.1.6 that S has a *least* upper bound. Call this point c. Now, since $a \in S$, we know that $c \in [a, b]$. By assumption, we know $a, b \in X$; hence, by Claim (2), we must conclude that $c \in X$ as well. Furthermore, since c is the least upper bound for S, we know that $H \cap (c - \epsilon, c] \neq \emptyset$ for all $\epsilon > 0$. (Otherwise, $c - \epsilon$ would be an upper bound for S to the left of c.) However, this means that any open set that contains c must also contain a member of H; and this implies that c is a limit point of H. Now, since H is closed by Exercise 1.10.1, we must conclude that $c \in H$.

We now turn attention to the set K. Let $\epsilon > 0$ and consider the set $[c, c+\epsilon)$. Either $c = b$ or c is to the left of b; hence, it follows that $[c, c+\epsilon) \cap [c, b] \neq \emptyset$. If $x \in [c, c+\epsilon) \cap [c, b]$, then clearly either $x = b$ or $a < x < b$. Consequently, we know by Claim (2) that $x \in X$. However, this forces us to conclude that $x \in K$ since x cannot be a member of H. Therefore, we know that $[c, c+\epsilon) \cap K \neq \emptyset$ for all $\epsilon > 0$. We must conclude that c is a limit point (and therefore a member of) K as well. This is impossible, however, since it implies that $c \in H \cap K$.

□

Theorem 1.10.3 tells us that the only connected subspaces of $\mathbb{R}$ under the usual topology are intervals, segments, rays, and $\mathbb{R}$ itself.

Exercise 1.10.4. Let (X, Ω) be a topological space. Prove that the following statements are equivalent.

1. X is connected
2. X has exactly two clopen subsets, namely $\emptyset$ and X.
3. There is no function $f : X \longrightarrow \{0, 1\}$ that is continuous relative to Ω and the discrete topology.

Exercise 1.10.5. Let (X, Ω) and (Y, Θ) be topological spaces, and suppose $f : X \longrightarrow Y$ is continuous relative to these topologies. If (X, Ω) is connected, use Exercise 1.10.4 to prove that $f(X)$ is connected as a subspace of (Y, Θ).

Exercise 1.10.6. Use Theorem 1.10.3 and 1.10.5 to give a very quick proof of the Intermediate Value Theorem.

Exercise 1.10.7. Let (X, Ω) be a topological space, and let $Y = \{0, 1\}$ be endowed with the discrete topology (see Exercise 1.2.2). Prove that (X, Ω) is connected if and only if there is a function $f : X \longrightarrow Y$ that is continuous relative to these topologies.

Exercise 1.10.8. Prove that any topological space that is homeomorphic to a connected topological space is itself connected.

A topological space is *totally disconnected* provided its only connected subsets are the singletons. Notice that in a totally disconnected space, singletons are clopen (hence every totally disconnected space is Hausdorff). It follows that every totally disconnected space has a basis of clopen sets (namely the set of singletons).

Theorem 1.10.9. *Let $\{(X_i, \Omega_i) : i \in I\}$ be a family of totally disconnected spaces. The set $\prod\{X_i : i \in I\}$ is totally disconnected under the product topology.*

Proof. Let $\mathbf{x}, \mathbf{y}$ be distinct members of $\prod\{X_i : i \in I\}$. There exist $i \in I$ such that $x_i = \pi_i(\mathbf{x}) \neq \pi_i(\mathbf{y}) = y_i$. Since (X_i, Ω_i) is totally disconnected, there exists a separation $A, B \in \Omega_i$ such that $x_i \in A$ and $y_i \in B$. Let U, V be opens in the product topology such that

$$\pi_j(U) = \begin{cases} A, & \text{if } j = i; \\ X_j, & \text{otherwise.} \end{cases} \qquad \pi_j(V) = \begin{cases} B, & \text{if } j = i; \\ X_j, & \text{otherwise.} \end{cases}$$

It is a routine matter to show that U, V form a separation of $\prod\{X_i : i \in I\}$ such that $\mathbf{x} \in U$ and $\mathbf{y} \in V$.

□

Exercise 1.10.10. Prove that every subspace of a totally disconnected topological space is itself totally disconnected. Give a counterexample to show that subspaces of connected spaces need not be connected.

Exercise 1.10.11. If (X, Ω) is a Hausdorff space with a clopen basis, prove that (X, Ω) is totally disconnected.

Exercise 1.10.12. If (X, Ω) is a totally disconnected, compact space, prove that (X, Ω) has a clopen basis. Hint: For each $x \in X$, let U be an open set containing x and work with the complement $Y = X - U$ to find a clopen V such that $x \in V \subseteq U$.)

Let I be an index set and for each $i \in I$ let $(X_i, \Omega_i) = (\{0, 1\}, \mathtt{Su}(\{0, 1\}))$. Let $\{0, 1\}^I = \prod\{X_i : i \in I\}$ and endow this set with the product topology. In light of Theorem 1.10.9, we know this space is totally disconnected.

Exercise 1.10.13. Prove that every compact, totally disconnected topological space is homeomorphic to a closed subspace of $\{0,1\}^I$ under the product topology for some index set I. Hint: Let $\mathcal{B} = \{B_i : i \in I\}$ be a clopen basis for the space. For each $i \in I$, let $f_i : X \longrightarrow \{0,1\}$ be defined by $f_i(x) = 1 \iff x \in B_i$, and let $\varphi : X \longrightarrow \{0,1\}^I$ be defined by $\pi_i(\varphi(x)) = f_i(x)$.

Theorem 1.10.14. *Let (X, Ω) be a topological space. Let (Y, Ω_Y) be a subspace and suppose that $A, B \in \Omega_Y$ such that $A \cap B = \emptyset$ and $A \cup B = Y$. The sets A and B form a disconnection of Y if and only if neither set contains a limit point of the other.*

Proof. Suppose that A and B form a disconnection of Y. We know that A and B are clopen. In particular, we know by Exercise **??** that $A = \overline{A} \cap Y$. Consequently, we know that $B \cap \overline{A} = \emptyset$; and we may conclude that B does not contain any limit points of A. By similar reasoning, A does not contain any limit points of B.

Conversely, suppose that A contains no limit points of B and B contains no limit points of A. We need to prove that A and B are clopen. Since $B \cap A = \emptyset$, it follows that $B \cap \overline{A} = \emptyset$ as well. Since we have assumed that $A \cup B = Y$, we see that

$$\overline{A} \cap Y = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = A$$

We may therefore conclude that A is closed. By similar arguments, we see that B is closed as well.

$\square$

Exercise 1.10.15. Let (X, Ω) be a topological space, and suppose that A and B form a disconnection of X. If (Y, Ω_Y) is any connected subspace of (X, Ω), prove that $Y \subseteq A$ or $Y \subseteq B$.

Exercise 1.10.16. Let $\mathcal{T} = \{(T_i, \Omega_i) : i \in I\}$ be a family of connected topological subspaces of a topological space (X, Ω). If $\bigcap\{T_i : i \in I\}$ is nonempty, prove that $\bigcup\{T_i : i \in I\}$ is a connected subspace of (X, Ω). (See Exercise 1.2.36.)

Exercise 1.10.17. Let (X, Ω) be a topological space, and suppose that (Y, Ω_Y) is a connected subspace of (X, Ω). If (Z, Ω_Z) is any subspace of (X, Ω) such that $Y \subseteq Z \subseteq \overline{Y}$, prove that Z is connected. (Use Exercise 1.10.15.)

Exercise 1.10.18. Show that $\mathbb{R}^\infty$ is not connected under the box topology. (See Exercise 1.6.26. Consider the sets A and B of bounded and unbounded sequences, respectively, and prove these are clopen in the box topology.)

Exercise 1.10.19. Let (X, Ω) be a topological space and let $x \in X$. Use Zorn's Lemma to prove that every connected subset of X is contained in a maximal connected subset of X. (The maximal connected subsets of X are called *connected components* of X.)

Exercise 1.10.20. Let (X, Ω) be a topological space. Prove that any connected component of X is closed, and show that the set of connected components forms a partition of X.

Exercise 1.10.21. Let (X, Ω) be a topological space. If X has only finitely many connected components, prove that they are clopen.

Exercise 1.10.22. Let (X, Ω) be a topological space. Prove that the following statements are equivalent.

1. (X, Ω) has a basis of connected sets
2. The connected components of every open set (under the subspace topology) are clopen .

Exercise 1.10.23. A topological space satisfying the conditions of Exercise 1.10.22 is *locally connected.* Prove that every countably compact, locally connected topological space has only finitely many connected components. (See Definition 1.5.1.)

Suppose that $\{(X_i, \Omega_i) : i \in I\}$ is a family of topological spaces. If the set $\prod\{X_i : i \in I\}$ is connected under the product topology, then Exercise 1.10.5 tells us that each factor (X_i, Ω_i) is connected. (This is because the projection maps π_i are continuous surjections.) The converse of this statement is also true; however, the proof is more challenging. We construct the proof in the following exercises.

Exercise 1.10.24. Suppose that $\{(X_i, \Omega_i) : i \in I\}$ is a family of topological spaces. Fix $i \in I$; and for all $j \in I - \{i\}$, fix $a_j \in X_j$. Define the function $f_i : X_i \longrightarrow \prod\{X_i : i \in I\}$ by $f_i(x) = \mathbf{x}$, where $\pi_i(\mathbf{x}) = x$ and $\pi_j(\mathbf{x}) = a_j$ when $j \neq i$. Use this map to prove that (X_i, Ω_i) is homeomorphic to the subspace $f_i(X_i)$. (This subspace is called the ith "strip" of the product.)

Exercise 1.10.25. Suppose that $\{(X_i, \Omega_i) : i \in I\}$ is a family of connected topological spaces and suppose $\mathbf{x}$ and $\mathbf{y}$ are members of $\prod\{X_i : i \in I\}$.

1. If $\mathbf{x}$ and $\mathbf{y}$ differ in only one coordinate, use the previous exercise and Exercise 1.10.5 to prove that they must lie in the same connected component under the product topology.

2. If $\mathbf{x}$ and $\mathbf{y}$ differ in only finitely many coordinates, use Part (1) to prove that they must lie in the same connected component under the product topology.

Exercise 1.10.26. Suppose that $\{(X_i, \Omega_i) : i \in I\}$ is a family of connected topological spaces and suppose there exist nonempty disjoint U, V in Ω_p which disconnect $\prod\{X_i : i \in I\}$. If $\mathbf{x}$ and $\mathbf{y}$ differ in only finitely many coordinates, prove that both must be in U or both must be in V.

We are now ready to construct a proof that the product of connected topological spaces is connected. We accomplish the proof by contradiction. Suppose that $\{(X_i, \Omega_i) : i \in I\}$ is a family of connected topological spaces and suppose there exist nonempty disjoint U, V in Ω_p which disconnect $\prod\{X_i : i \in I\}$. Since U is nonempty, we know there exists a nonempty basic open $W \subseteq U$. By assumption, there are only finitely many $i \in I$ such that $\pi_i(W) \neq X_i$. Let $i_1, ..., i_n$ denote these indices. Let $\mathbf{x} \in \prod\{X_i : i \in I\}$. There exist $\mathbf{y} \in W$ which differs from $\mathbf{x}$ in at most finitely many coordinates (some subset of $\{i_1, ..., i_n\}$). Since $\mathbf{x}$ and $\mathbf{y}$ must lie in the same connected component, they must both be in U (otherwise the component can be disconnected). This implies that $V = \emptyset$ — contrary to assumption.

Definition 1.10.27. Let (X, Ω) be a topological space, and let $[0, 1]$ be endowed with the usual topology. A continuous function $f : [0, 1] \longrightarrow X$ is called a *path* joining $f(0)$ to $f(1)$. We say that (X, Ω) is *path connected* provided every distinct pair of points in X can be joined by a path.

Exercise 1.10.28. Let (X, Ω) be a topological space. Prove that the following statements are equivalent.

1. (X, Ω) is path connected.
2. There exist $a \in X$ that can be joined to every point in X.

Exercise 1.10.29. Let (X, Ω) be a path connected space. If (Y, Θ) is any topological space and $f : X \longrightarrow Y$ is continuous relative to these topologies, prove that $f(X)$ is a path connected subspace of (Y, Θ).

Exercise 1.10.30. Prove that every path connected topological space is also connected.

The converse of Exercise 1.10.30 is false — there exist connected topological spaces that are not path connected. One famous example is the so-called "topologist's sine". The topologist's sine is the closure in $\mathbb{R}^2$ (under the usual product topology) of the set

$$G = \left\{ \left(x, \sin\left(\frac{\pi}{x}\right) \right) : x \in (0,1] \right\}$$

If we assume that the function $g : (0,1] \longrightarrow [-1,1]$ is continuous relative to the usual topology, then Theorem 1.10.3, Exercise 1.10.29, and Exercise 1.10.17 tell us that $\overline{G}$ is connected as a subspace of $\mathbb{R}^2$. However, it is impossible to construct a continuous function $f : [0,1] \longrightarrow \bar{G}$ such that $f(0) = (0,0)$ and $f(1) = (1,0)$. This is probably intuitively obvious, but its proof requires a little work.

To begin, suppose that such a function f does exist. Let $X = \{0\} \times [-1,1]\}$ (this set is the boundary for $\overline{G}$ and is therefore closed). By assumption, $f(0) = (0,0)$, so we know that $f([0,1]) \cap X \neq \emptyset$. Let $Y = f^{-1}(X)$. Since f is continuous, Y is a closed subset of $[0,1]$ and therefore must have a largest member since it is nonempty. Let b be the largest member of Y and consider the restriction $f' : [b,1] \longrightarrow \overline{G}$. Let $f'(t) = (x(t), y(t))$ be a parametrization of f'. Since f' is continuous, we know that x and y are continuous as well.

Now, for each positive integer n, note that $x(b + 1/n) > 0$, since b was chosen to be the largest member of $[0,1]$ mapped into X. Pick u so that $0 < u < x(b + 1/n)$ and $sin(\pi/u) = (-1)^n$. (The Intermediate Value Theorem guarantees this is possible.) The Intermediate Value Theorem also guarantees that there exist t_n such that $0 < t_n < 1/n$ and $x(t_n) = u$. Now, the sequence $\{t_n : n \in \mathbb{Z}^+\}$ converges to 0, but the sequence $\{y(t_n) : n \in \mathbb{Z}^+\}$ does not converge at all. This contradicts Theorem 1.5.26; hence, we must conclude that no such function f exists.

Exercise 1.10.31. Let (X, Ω) be a topological space. Prove that the following statements are equivalent.

1. (X, Ω) is path connected.

2. (X, Ω) is connected and every member of X is contained in a path connected open set.

Let (X, Ω) be a topological space. If $A \subseteq X$ is path connected, then Zorn's Lemma can be used to prove that A is contained in a maximal path connected subset of X. A maximal path connected subset of X is called a *path component* of X. Like connected components, the path connected components of X form a partition of X. Connected components of X are always closed, but this is not the case with path components. The topologist's sine provides a counterexample. The set G is the continuous image of the set $(0,1]$ and is therefore path connected. If A is any path component of $\overline{G}$ that contains G, then we know that $\overline{A} = \overline{G}$; hence, $\overline{A}$ is not path connected. Thus, A cannot be closed. (It can be shown that $A = G$.)

1.11 Quotient Spaces

The various product topologies we introduced on families of topological spaces provide us with ways to build up spaces from simpler ones. To some extent, it is possible to go in the other direction — we can create topologies from existing ones which are in some ways "simpler" than the ones we started with. This can help understand properties of the original space, and it can lead to entirely new constructions.

Definition 1.11.1. Let (X, Ω) be a topological space, and suppose that X_q is a partition of X. For each $x \in X$, let $[x]$ denote the member of X_q that contains x. The *quotient topology* relative to Ω is the finest topology on X_q such that the mapping $\nu : X \longrightarrow [x]$ is continuous.

The function ν is called the *natural* map; you are probably familiar with it from abstract algebra or ring theory where it is defined the same way. If we let Ω_q denote the quotient topology on X_q, then $U \in \Omega_q$ if and only if $\nu^{-1}(U) \in \Omega$. While the definition of quotient topology is straightforward, it is generally not easy to determine exactly what the quotient topology for a given space and partition will look like. Let's start by looking at a simple example to get some idea how the process works.

Let $X = \{a, b, c, d, e\}$ and let $\Omega = \{\emptyset, X, \{a, e\}, \{a\}, \{a, b, c, e\}\}$. As a partition on X, let $X_q = \{\{a, e\}, \{b, c\}, \{d\}\}$. The natural map is now determined; it is given by the following assignment.

$$\nu(a) = \{a, e\} = \nu(e) \qquad \nu(b) = \{b, c\} = \nu(c) \qquad \nu(d) = \{d\}$$

The quotient topology is now determined as well, although it is not obvious. There are eight subsets of X_q, namely

$$\mathtt{Su}(X_q) = \{\emptyset, X_q, \{[a]\}\{[b]\}, \{[d]\}, \{[a], [b]\}, \{[b], [d]\}, \{[a], [b], [d]\}\}$$

It should be clear that $\emptyset$ and X_q are members of Ω_q, since $\nu^{-1}(\emptyset) = \emptyset$ and $\nu^{-1}(X_q) = X$, both of which are open in Ω. Are there any other members of the powerset whose inverse image is open under ν? Note that $\nu^{-1}(\{[a]\}) = \{a, e\}$, so the singleton $\{[a]\}$ will be in Ω_q. Likewise, $\nu^{-1}(\{[a], [b]\}) = \{a, b, c, e\}$, so this set will be in Ω_q. There are no others. Consequently, the quotient topology will be

$$\Omega_q = \{\emptyset, \{[a]\}, \{[a], [b]\}, X_q\}$$

The real purpose of forming quotient topologies is the same as the purpose of forming quotient groups or rings — by "identifying" elements through a partition, the space is to some extent simplified. In topology, this process of identification can have concrete and surprising effects. Consider, for example, the unit square

$$S = [0,1] \times [0,1]$$

as a subspace of the usual product topology on $\mathbb{R}^2$. In this case, the topology Ω is generated by the family $\mathcal{B}$ consisting of the empty set and all possible products of pairs of segments and half-open intervals (a,b), $[0,a)$, and $(a,1]$ where $0 < a, b < 1$. Consider the partition S_q which identifies only the vertical boundaries of the square and leaves every other point equal to itself:

$$\begin{aligned} S_q &= \{\{(a,b)\} : 0 < a, b < 1\} \cup \{\{a,0\} : 0 < a < 1\} \\ &\cup \{\{(1,a)\} : 0 < a < 1\} \cup \{\{(0,a),(1,b) : 0 \leq a, b \leq 1\}\} \end{aligned}$$

In physical terms, the effect of the partition S_q is to "weld together" the vertical sides of the square, leaving the interior and horizontal boundaries of the square intact. This transforms the unit square into a cylinder whose surface is the original interior of the square.

Now, we would like for the quotient topology on S_q to be homeomorphic to the subspace topology in $\mathbb{R}^3$ which represents the cylinder. Of course, it is not particularly easy to describe the subspace topology for the cylinder. One way to describe it is to note that it is generated by the family consisting of the intersection of all open spheres with the cylinder; however, the resulting surfaces are rather complicated. To complicate matters further, it is not obvious what the opens of the quotient topology will be. We will deal with this problem the same way we dealt with it in group and ring theory — we will develop tools which help us identify spaces homeomorphic to the quotient space without having to compare the topologies open by open.

Let (X,Ω) and (Y,Θ) be topological spaces and let $f : X \longrightarrow Y$. We say that f is *open* provided $U \in \Omega$ implies $f(U) \in \Theta$. We say that f *preserves open sets*. Similarly, we say that f is *closed* provided it preserves closed sets. There is no continuity assumption in our definition of open and closed functions, although some texts do make this assumption.

Exercise 1.11.2. Let $X = [0,1] \cup [2,3]$ and $Y = [0,2]$ be endowed with the usual subspace topology on $\mathbb{R}$. Consider the function

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1]; \\ x-1, & \text{if } x \in [2,3]. \end{cases}$$

Show that f is a closed map but is not an open map.

Exercise 1.11.3. Consider $\mathbb{R}^2$ under the usual product topology.

1. Prove that the projection map $\pi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an open map.
2. Prove that the set $C = \{(a,b) \in \mathbb{R}^2 : ab = 1\}$ is closed.

3. Prove that $\pi_1(C)$ is not closed in $\mathbb{R}$.

Definition 1.11.4. Let (X, Ω) and (Y, Θ) be topological spaces and let $f : X \longrightarrow Y$. We say that f is a *quotient map* provided f is continuous, f is surjective, and $f^{-1}(V) \in \Omega$ if and only if $V \in \Theta$.

Exercise 1.11.5. Prove that the composition of two quotient maps is also a quotient map.

Exercise 1.11.6. Prove that a bijective quotient map is a homeomorphism.

Let (X, Ω) be a topological space, and let X_q by a partition of X. Notice that endowing X_q with the quotient topology makes the natural map a quotient map. You may also recall that in group and ring theory, every surjective homomorphism gives rise to a quotient algebra (induced by its kernel). Our definition of quotient map will lead us to a similar conclusion for topologies.

Exercise 1.11.7. Let (X, Ω) and (Y, Θ) be topological spaces and suppose that $f : X \longrightarrow Y$ is surjective and continuous relative to these topologies. If f is an open or a closed map, prove that f is a quotient map.

Exercise 1.11.8. Consider $\mathbb{R}^2$ under the usual product topology and let $A = \{(x, y) : x \geq 0 \,\text{or}\, y = 0\}$ be endowed with the usual subspace topology. If $q : A \longrightarrow \mathbb{R}$ is the restriction to A of the projection map π_1, show that q is a quotient map that is neither open nor closed.

We see that the notion of surjective, continuous "open map" is stronger than the notion of "quotient map." While quotient maps need not be open, it is true that quotient maps preserve *some* open sets. Let X and Y be sets and suppose that $f : X \longrightarrow Y$ is a surjection. A subset V of X is *saturated* provided it is the inverse image of some subset of Y. In other words, there exists some $A \subseteq Y$ such that

$$V = f^{-1}(A) = \bigcup \{f^{-1}(y) : y \in A\}$$

Exercise 1.11.9. Let (X, Ω) and (Y, Θ) be topological spaces and suppose that $f : X \longrightarrow Y$ is a continuous surjection. Prove that the following statements are equivalent.

1. f is a quotient map
2. f maps saturated open sets to open sets
3. f maps saturated closed sets to closed sets.

Notice that the functions in Exercises 1.11.2 and 1.11.3 are quotient maps.

Exercise 1.11.10. Suppose X and Y are sets and suppose that $f : X \longrightarrow Y$ is any function. If $\mathcal{F} \subseteq \mathtt{Su}(Y)$, then prove the following.

1. $f^{-1}\left(\bigcup \mathcal{F}\right)) = \bigcup\{f^{-1}(A) : A \in \mathcal{F}\}$
2. $f^{-1}(A_1 \cap ... \cap A_n) = \bigcap\{f^{-1}(A_1), ..., f^{-1}(A_n)\}$ for any $A_1, ..., A_n \in \mathcal{F}$.

Exercise 1.11.11. Let (X, Ω) be a topological space and let Y be any set. Suppose that $f : X \longrightarrow Y$ is any surjective function and consider the family

$$\Theta_f = \{U \in \mathtt{Su}(A) : f^{-1}(U) \in \Omega\}$$

Use Exercise 1.11.10 to show that Θ_f is a topology on A then show that f is a quotient map with respect to this topology. (This is called the topology *induced* by the funtion f.)

Exercise 1.11.12. Suppose that (X, Ω), and suppose that $f : X \longrightarrow Y$ is a surjection. Let Θ_f be the topology on Y induced by the function f, and suppose $X_q = \{f^{-1}(a) : a \in Y\}$.

1. Explain why X_q is a partition of X.
2. Assume that (X_q, Θ_q) is the quotient topology, and assume that $\nu : X \longrightarrow X_q$ is the natural map. Define $\varphi : X_q \longrightarrow Y$ by $\varphi(f^{-1}(a)) = a$. Prove that φ is a bijection.
3. Explain why $\varphi \circ \nu = f$.
4. For each $a \in Y$, show that $f^{-1}(a) = \nu^{-1}(\varphi^{-1}(a))$.
5. Use Exercise 1.11.10 to show that for all $U \in \Theta_f$, we have $f^{-1}(U) = \nu^{-1}(\varphi^{-1}(U))$.
6. Prove that φ is a homeomorphism.

The previous exercises show two things. First, every surjection f from a topological space (X, Ω) to a set Y induces a "natural" topology on Y, namely one over which the surjection is a quotient map. Second, this "natural" topology is homeomorphic to a quotient space on the original space. In light of this, we usually refer to this topology as the quotient topology on X induced by the function f.

Exercise 1.11.13. Let $\mathbb{R}$ be endowed with the usual topology and let $Y = \{a, b, c\}$. Define a surjection $f : \mathbb{R} \longrightarrow Y$ by

$$f(x) = \begin{cases} a, & \text{if } x > 0; \\ b, & \text{if } x < 0; \\ c, & \text{if } x = 0. \end{cases}$$

1. Determine the topology induced by f on the set Y.
2. What is the partition of $\mathbb{R}$ induced by f.
3. Are there open sets in $\mathbb{R}$ that are not saturated?

Exercise 1.11.14. Consider the disk $X = \{(x,y) : x^2 + y^2 \leq 1\}$ endowed with the usual subspace topology. Let $Y = \{(a,b,c) : a^2 + b^2 + c^2 = 1\}$. Note that Y is the surface of the unit sphere in $\mathbb{R}^3$. Construct a surjection $f : X \longrightarrow Y$ that maps the boundary of X to the north pole $(0,0,1)$ and maps the interior of the disk to the remainder of the sphere as a bijection. Hint: Let $f(0,0) = (0,0,-1)$. Represent the interior of the disk in polar coordinates, then use spherical coordinates to map circles of radius r $(0 < r < 1)$ to circular traces on the sphere, from the bottom to the top.

Consider the subspace X and the set Y in the previous exercise. The surjection f induces a topology on the sphere Y which is homeomorphic to a particular quotient topology on the disk X. By construction, the underlying partition for this quotient topology is

$$\begin{aligned} X_q &= \{f^{-1}(a,b,c) : (a,b,c) \in Y\} \\ &= \{f^{-1}(a,b,c) : (a,b,c) \neq (0,0,1)\} \cup \{f^{-1}(0,0,1)\} \\ &= \{\{x,y)\} : x^2 + y^2 < 1\} \cup \{\{(x,y) : x^2 + y^2 = 1\}\} \end{aligned}$$

Notice that this partition "collapses" the boundary of the disk onto the north pole of the sphere, while the interior of the disk "wraps" into the remainder of the sphere, much like a paper lantern.

The way the function f is created in Exercise 1.11.14, we see that any open disk interior to X is mapped to an open disk on the surface of the sphere that does not include the north pole. Any annulus $A_r = D - \{(x,y) : x^2 + y^2 \leq r\}$ (where $0 < r < 1$) is also open in the disk and is mapped to an open disk on the surface of the sphere that is centered on the north pole.

Exercise 1.11.15. Let (X, Ω) be a topological space, and suppose that (X_q, Ω_q) is a quotient space for X with natural map ν. Show that there exists a set Y and a surjection $f : X \longrightarrow Y$ such that $X_q = \{f^{-1}(a) : a \in Y\}$. (Hint: Let Y be the set consisting of exactly one element chosen from each equivalence class in X_q.)

The previous exercise completes the picture relating quotient maps and quotient spaces. If (X_q, Ω_q) is any quotient space on a topological space (X, Ω), then there is some set Y and a surjection $f : X \longrightarrow Y$ such that (X_q, Ω_q) is homeomorphic to the topological space on Y induced by f. In other words, up to homeomorphism, we can create quotient spaces by forming surjections. Consequently, as in ring or group theory, when the need for quotient spaces arises, our main concern is finding spaces that are *homeomorphic* to the quotient space we need, rather than trying to construct that space directly.

Exercise 1.11.16. Let (X, Ω) be a topological space. Prove that a quotient space (X_q, Ω_q) is T_1 if and only if each member of X_q is a closed subset of X.

Exercise 1.11.17. Let (X, Ω), (Y, Θ), and (Z, Γ) be topological spaces and suppose $q : X \longrightarrow Y$ is a quotient map. Suppose further that $g : X \longrightarrow Z$ is constant on each set $q^{-1}(a)$ such that $a \in Y$. Prove the following statements.

1. There exists a function $f : Y \longrightarrow Z$ such that $f \circ q = g$.
2. The function f is continuous relative to Θ and Γ if and only if g is continuous relative to Ω and Γ.
3. The function f is a quotient map if and only if g is a quotient map.

It may be worth noting that the results of Exercise 1.11.12 constitute a special case of the previous exercise.

The term "quotient space" is inspired by quotients in groups and rings; and we have seen that constructing quotient spaces bears some resemblance to constructing quotient groups or rings. There is a deeper relation that briefly explore.

Definition 1.11.18. Suppose that $\mathcal{G} = (G, *, ()^{-1})$ is a group. We say that $\mathcal{G}$ is a *topological group* provided there is a T_1 topology Ω on G which makes the group multiplication and inversion operations continuous. In other words, a topological group is a quadruple $\mathcal{G} = (G, \Omega, *, ()^{-1})$ such that

1. $(G, *, ()^{-1})$ is a group,
2. (G, Ω) is a T_1 topological space,
3. the operation $* : G \times G \longrightarrow G$ is continuous relative to Ω and the product topology on $G \times G$ built from Ω
4. the operation $()^{-1} : G \longrightarrow G$ is continuous relative to Ω.

When there is a topology on a group which makes it a topological group, we say that the group forms a topological group *relative* to this topology. There may be many topologies which make a particular group into a topological group.

Exercise 1.11.19. Let $\mathcal{G} = (G, *, ()^{-1})$ be a group and suppose that (G, Ω) is a T_1 topological space. Prove that $\mathcal{G}$ is a topological group relative to Ω if and only if the mapping $f : G \times G \longrightarrow G$ defined by $f(x, y) = x * y^{-1}$ is continuous.

Exercise 1.11.20. Show that the real numbers under addition forms a topological group relative to the usual topology.

Exercise 1.11.21. Suppose $\mathcal{G} = (G, \Omega, *, ()^{-1})$ is a topological group, and suppose that H is a subspace of (G, Ω). If H is also a subgroup of $\mathcal{G}$, prove that both H and $\overline{H}$ are topological groups relative to the subspace topology.

Exercise 1.11.22. Suppose $\mathcal{G} = (G, \Omega, *, ()^{-1})$ is a topological group and let $a \in G$. Show that the maps $f_a : G \longrightarrow G$ and $h_a : G \longrightarrow G$ defined by $f_a(x) = a * x$ and $h_a(x) = x * a$ are homeomorphisms.

Let $\mathcal{G} = (G, *, ()^{-1})$ be a group and let H be a subgroup of $\mathcal{G}$. For $a \in G$, it is traditional to let

$$aH = \{a * h : h \in H\} \qquad Ha = \{h * a : h \in H\}$$

denote the left and right *cosets* of H, respectively, that are generated by a. Recall that the families $\mathcal{L} = \{aH : a \in G\}$ and $\mathcal{R} = \{Ha : a \in G\}$ both form partitions of G; furthermore, there is a bijective correspondence between aH and bH for all $a, b \in G$. (There is also a bijective correspondence between Ha and Hb for all $a, b \in G$.)

Suppose that $\mathcal{G} = (G, \Omega, *, ()^{-1})$ is a topological group. The "natural" mapping $\nu : G \longrightarrow \mathcal{L}$ defined by $\nu(x) = xH$ is clearly a surjection; hence we may endow $\mathcal{L}$ with the quotient topology induced by ν by Exercise 1.11.12. The following exercises will assume that this has been done.

Exercise 1.11.23. Let $\mathcal{G} = (G, \Omega, *, ()^{-1})$ be a topological group and let H be a subgroup of $\mathcal{G}$.

1. For each $a \in G$, show that the mapping f_a defined in Exercise 1.11.22 induces a homeomorphism between $\mathcal{L}$ and itself. (Consider the mapping $g : \mathcal{L} \longrightarrow \mathcal{L}$ defined by $g_a(xH) = f_a(x)H$.)
2. If H is closed in (G, Ω), prove that $\mathcal{L}$ is a T_1 space.
3. Show that the natural map ν is open.

Let $\mathcal{G} = (G, *, ()^{-1})$ be a group and let H be a subgroup of $\mathcal{G}$. Recall that H is *normal* in $\mathcal{G}$ provided $aH = Ha$ for all $a \in G$. A subgroup H of $\mathcal{G}$ is normal if and only if the relation $aH \otimes bH = (a * b)H$ defines a binary operation on $\mathcal{L}$. When this is the case, $\mathcal{L}$ forms a group with respect to $\otimes$. Inversion is defined in the obvious way: $(aH)^{-1} = a^{-1}H$. This group is called the *quotient* group of $\mathcal{G}$ relative to H (or simply $\mathcal{G}$ mod H) and is often denoted by $\mathcal{G}/H$.

Exercise 1.11.24. Let $\mathcal{G} = (G, \Omega, *, ()^{-1})$ be a topological group and let H be a normal subgroup of $\mathcal{G}$. Prove that $\mathcal{G}/H$ is a topological group.

Let $\mathcal{G} = (G, *, ()^{-1})$ be a group and let $X, Y \subseteq G$. We will let $X^{-1} = \{x^{-1} : x \in X\}$ and let $XY = \{x * y : x \in X, y \in Y\}$. If $\mathcal{G} = (G, \Omega, *, ()^{-1})$ is a topological group, then a neighborhood V of the identity element is said to be *symmetric* provided $V = V^{-1}$. If U is any neighborhood of the identity, then it is easy to see that UU^{-1} is symmetric.

Exercise 1.11.25. Let $\mathcal{G} = (G, \Omega, *, ()^{-1})$ be a topological group. If U is a neighborhood of the identity element, prove there exists a symmetric neighborhood V of the identity such that $VV \subset U$.

Exercise 1.11.26. Let $\mathcal{G} = (G, \Omega, *, ()^{-1})$ be a topological group. If $x \neq y$ in G, show that there exists a neighborhood V of the identity element such that $V\{x\} \cap V\{y\} = \emptyset$. (Consequently, all topological groups are Hausdorff spaces.)

Exercise 1.11.27. Let $\mathcal{G} = (G, \Omega, *, ()^{-1})$ be a topological group. If A is a closed subset of G and $x \in G - A$, show that there exists a neighborhood V of the identity element such that $VA \cap V\{x\} = \emptyset$. (Consequently, all topological groups are regular.)