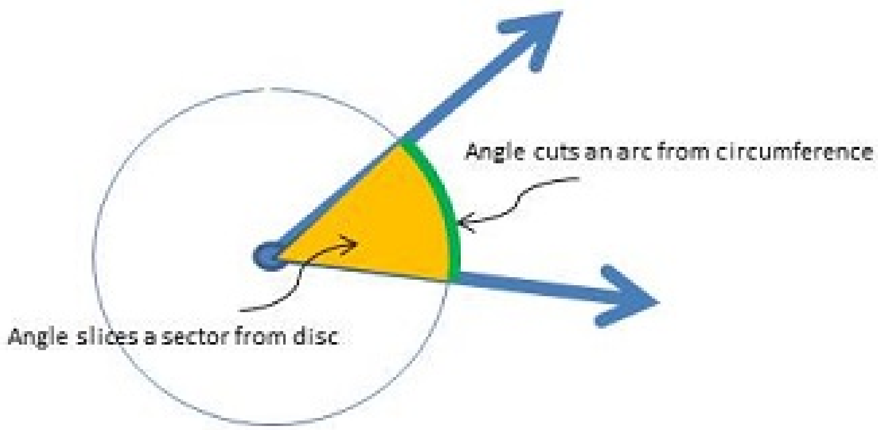
TRIGONOMETRY

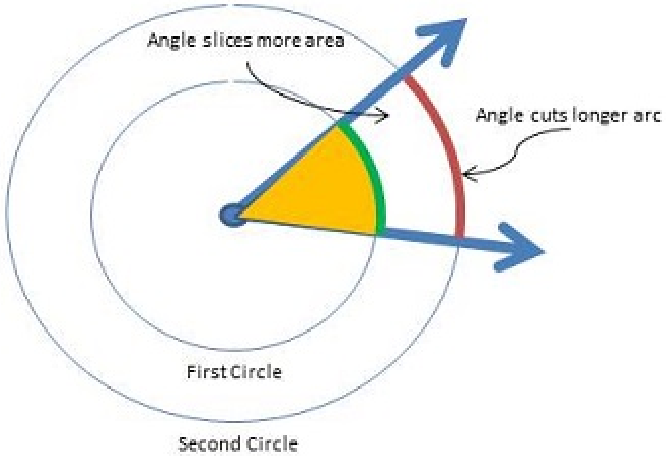
1 Investigation 1 — Measuring Angles

An *angle* is formed by two rays that have the same endpoint. This endpoint is called the *vertex* of the angle. Every pair of rays that have the same endpoint actually make *two* angles, as the diagram below shows.

We have two standard ways to measure the "*openness*" of angles, and to understand something about these ways, it is helpful to let the vertex of your angle be the center of a circle. It does not matter what the radius of this circle is.

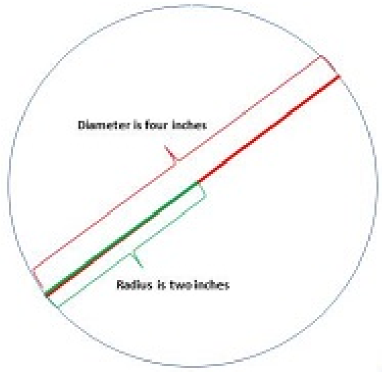


The angle shown above slices a sector from the interior of the circle, and it also cuts an arc from the circumference of the circle. We could measure the area of this sector, or we could measure the length of this arc. Therefore, we might consider using either as a way to measure the "openness" of the angle. However, there is a big problem with this approach — what if we use a *different* circle instead?

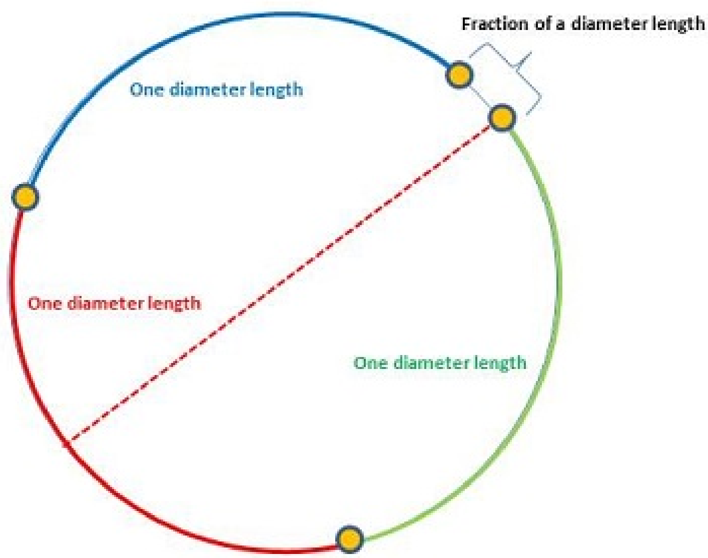


As the figure above shows, if we use a bigger circle, the *same* angle slices a larger sector and cuts a longer arc. We will have to adopt a more subtle approach in order to measure the "openness" of an angle.

Long ago, scientists decided that the circumference length of a circle would be measured in terms of the circle's *radius*. They chose to do this because there is a peculiar relationship between the circumference length and the radius length of any circle. The diagram below shows a circle with a radius of two inches.



How many *diameter lengths* can we fit around the circumference of this circle? To answer this, imagine we take pieces of flexible wire that are each four inches long and bend them end-to-end around the circumference of the circle. Here is what we come up with.



The diagram shows that we can fit three full diameter lengths around the circumference of the circle, along with a *fraction* of a fourth diameter length. The value of this fraction has been estimated to billions of decimal places — it is approximately 0.14159. Consequently, we can fit approximately 3.14159 diameter lengths around the circumference of this circle.

It turns out that no matter what circle we use, we can always fit the same number of diameter lengths around that circle's circumference. Mathematicians in the late 1800's proved that it is not possible to determine the decimal form of this number exactly. The special symbol π (from the Greek word for "perimeter") was adopted to represent this number.

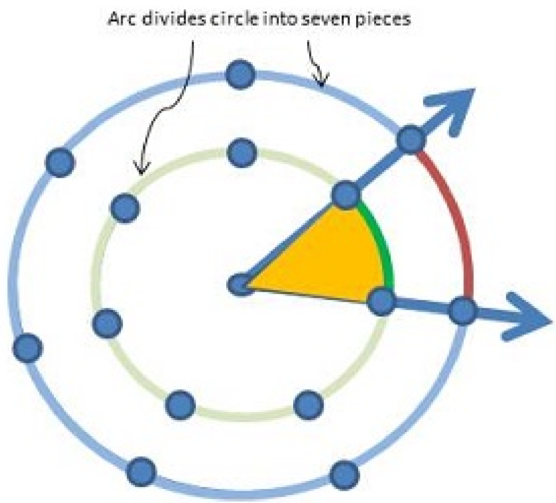
π represents the number of diameter lengths that can be fit around a circle's circumference

Now, since the radius of a circle is half its diameter, we can fit 2π radius lengths around the circle circumference. This gives us a simple way of relating the length of a circle's circumference to the length of its radius. If we let R represent the radius length of the circle and let C represent its circumference length, then

$$C = (2\pi) \cdot R$$

Because scientists have chosen to measure the circumference length of a circle using radius lengths, the number 2π now appears almost everywhere in science.

Let's return to the angle we centered in two circles above. The rays will cut (*subtend*) an arc from the circumference of each circle. The length of this arc will be some fraction of the circumference length, and the *fraction* of the circumference length the arc represents does not depend on the circle we use.



Although the arcs cut by this angle from each circle have different lengths, they are both one-seventh the circumference lengths of the circles they are cut from. This observation points out something critical about the relationship between the circles we draw and the arcs cut by the angle from each one:

Fundamental Property of Angles and Arcs

- If the vertex of an angle is at the center of two circles, the arc lengths cut by the angle will be the same fraction of each circle's circumference length.

Suppose we let R be the radius of either circle in the diagram above (measured in inches, let's say). The length S of the arc cut by this angle from either circle is one-seventh the circumference length, so we know

$$S = \frac{1}{7} \cdot 2\pi R \text{ inches} \quad \Longrightarrow \quad S = \left(\frac{2\pi}{7}\right) \cdot R \text{ inches}$$

The fraction $\frac{2\pi}{7}$ is the *radian measure* of the angle. Note that this fraction has no units associated with it since both S and R are measured in inches.

The number $\frac{2\pi}{7}$ is really just the number of radius lengths that fit on the arc cut by the rays of the angle.

RADIAN MEASURE OF AN ANGLE

Suppose we have an angle centered in a circle. There are two equivalent ways to understand the radian meausre of this angle.

- The *radian measure* of this angle is *the number of radius lengths* that fit on the arc cut from the circle's circumference by the rays that determine the angle.

- The *radian measure* of this angle is *the percentage of the radius length* represented by the arc cut from the circle's circumference by the rays that determine the angle.

These two ways of defining the radian measure of an angle really say the same thing. In the previous example, the percentage of the radius R represented by the arc length S is just the fraction S/R. Now, since we know

$$S = \left(\frac{2\pi}{7}\right) \cdot R \text{ inches}$$

we also know that the percentage of the radius length represented by S is the fraction

$$\frac{S}{R} = \frac{2\pi}{7}$$

Problem 1. Suppose the vertex of an angle is the center of a circle having a two foot radius. If the arc cut from this circle by the angle is five inches long, what is the radian measure of this angle?

Problem 2. Suppose the radian measure of an angle is $\theta = 2.31$. If the vertex of this angle is the center of a circle with a three-meter radius, how long is the arc cut by this angle?

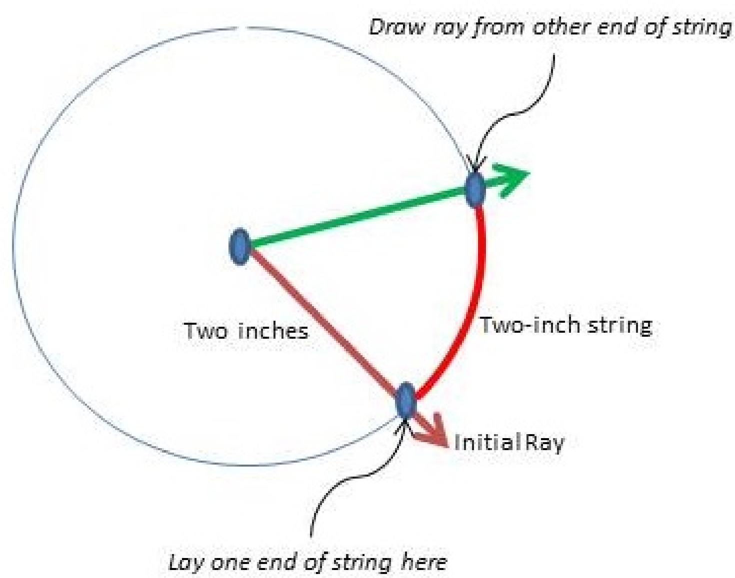
Problem 3. Suppose the radian measure of an angle is $\theta = \dfrac{7\pi}{8}$. If the vertex of this angle is the center of a circle, and the angle cuts an arc three feet long from this circle, what is the length of the radius?

Problem 4. The vertex of an angle is at the center of a circle whose radius is measured in miles, and the arc it cuts from this circle is exactly one-fourth the circle's circumference. What is the radian measure of this angle?

The radian measure of an angle is unitless. However, it is sometimes helpful to have some type of unit to work with when we are dealing with radian measure, so we invent a special unit just for this purpose. One *radian* is defined to be the angle required to cut an arc from any circle that is exactly equal in length to the circle's radius. This special angle is called a `rad` — the arc cut from any circle by a `rad` is equal to one radius length. The radian measure of any angle can be found by counting the number of `rad`'s enclosed between the two rays that define the angle.

Example 1 *What does a one- radian angle look like?*

Solution. To see what such an angle looks like, let's work with a circle whose radius is two inches. When its vertex is the center of this circle, one radian will cut an arc from the circle that is exactly two inches long. We can draw one radius for this circle, then cut a two-inch piece of string and lay it on the perimeter of the circle so that one end lies at point where the radius we drew intersects the circle. We then draw another ray from the other end of the string back to the center. The resulting "gap"is a one-radian angle.



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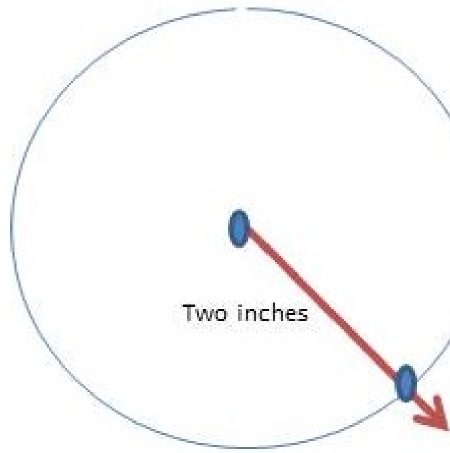
Problem 5. What does it mean to say that an angle has a measure of 2.67 `rad`?

Problem 6. Assume the circle below has a radius of two inches. Using the initial ray given and a piece of string, draw an angle whose measure is exactly 3 `rad`.



DEGREE MEASURE

Degree measure is another, much older way to measure angles. It is not used much in mathematics, physics, or engineering for a number of reasons, most of which have to do with calculus. It is still widely used in navigation, surveying, and meteorology however; and it is therefore worth mentioning. In degree measure, the circumference of a circle is is divided into 360 equal segments, and the number of these segments enclosed between the rays that define the angle is called the *degree measure* of that angle. To measure an angle in degrees, we imagine its vertex is at the center of a circle (of any radius), and we simply count the number of degrees enclosed between the rays that define the angle. It is traditional to use a superscipted circle on the degree measurement to indicate that the angle measurement is in degrees. For example

Angle measure of 38.92 degrees is written 38.92°

Problem 7. Consider an angle whose degree measure is 120°.

Part (a). What percentage of 360° is the degree measure for this angle?

Part (b). Explain why the radian measure of this angle must be the same percentage of 2π `rad`.

Part (c). What is the radian measure for this angle?

Problem 8. Use the method of the previous exercise to fill in the table below.

Degree Measure of Angle	Percentage of 360°	Radian Measure of Angle
0°		
15°		
30°		

Problem 9. Suppose an angle has radian measure 2.25 `rad`.

Part (a). What percentage of 2π `rad` is the radian measure of this angle?

Part (b). Explain why the degree measure of this angle must be the same percentage of 360°.

Part (c). What is the radian measure for this angle?

Supplemental Homework Problems

1. Explain what it means in terms of arc length and radius for an angle to have radian measure

   (a) 1 `rad` (b) 2.1 `rad` (c) π `rad` (d) $\frac{\pi}{6}$ `rad`

2. Convert the following angle measures from degree measure to radian measure.

   (a) 37° (b) 310° (c) 25.75° (d) 1°

3. Convert the following angle measures from radian measure to degree measure.

   (a) 1 `rad` (b) 3 `rad` (c) π `rad` (d) $\frac{\pi}{4}$ `rad`

4. Three angles each have their vertex at the center of a circle having a radius of twenty-one inches. The measures of these angles are given below. Determine the length in inches of the arc cut from the circle by each angle.

   (a) 1 `rad` (b) 3 `rad` (c) $\frac{3\pi}{2}$ `rad` (d) 120°

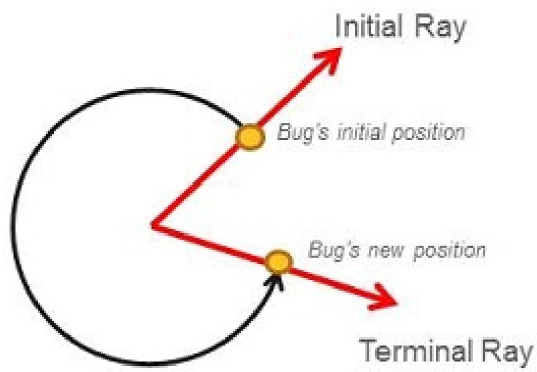
5. Mildred walks four hundred feet around a circular track having a one-hundred foot radius. What is the radian measure of the angle Mildred creates as she walks from her starting point to her finishing point? What is the degree measure of this angle?

6. Jose boards a ferris wheel having a radius of fifty feet. The ferris wheel starts turning and Jose rotates through a $217°$ angle before the wheel stops to allow more passengers to get on. How far in feet did Jose travel around the wheel?

2 Investigation 2 — A Second Look at Angle Measure

In the sciences, angles are often used to represent rotations. Angles used in this way are sometimes called *rotation* angles. In this investigation, we will explore an example of how rotation angles are interpreted and used.

A bug lands on the tip of one blade on a ceiling fan having a two-foot radius. Jade turns on the fan, and it begins to rotate slowly counterclockwise at a constant speed. As the fan blades rotate, the bug will move from its initial loctaion counterclockwise along the circumference of a two-foot radius circle. Imagine a ray extending from the center of the fan through the bug's initial position. If we took a photo of the fan (freezing the motion) and drew a ray from the center of the fan to the bug's new position, these two rays would form an angle.



The ray passing through the bug's initial position is called the *initial ray* of the angle formed. The ray passing through the bug's new position is called the *terminal ray* of the angle formed. The bug's path is shown as the directed arc in the diagram above. (This is called a *direction arc* for the angle). The arrow on a direction arc always points from the initial ray to the terminal ray in the direction of the rotation.

Problem 1. Suppose the bug travels nine feet around the circumference of the circle from its initial position to its new position.. What is the radian measure of the angle in the diagram above?

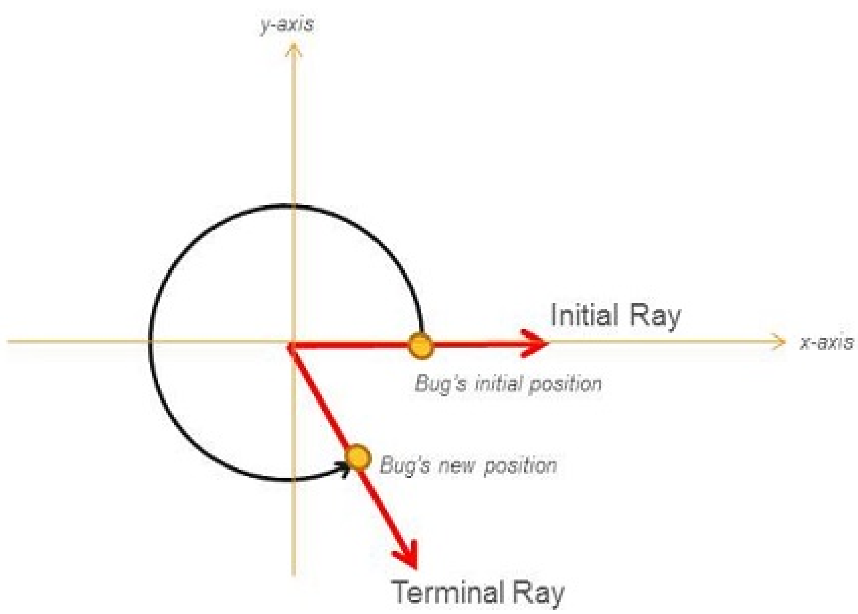
Problem 2. Suppose a wad of gum is stuck to the same fan blade one foot from the center of the fan. Through what distance will the wad of gum travel as the bug goes from its initial position to its new position?

Problem 3. The *angular speed* of the bug is defined to be the number of radians (radius lengths) the bug travels in one unit of time. (The units of angular speed are `rad`'s per unit of time.) If it took the bug three seconds to travel from its initial position to its new position, what is its angular speed?

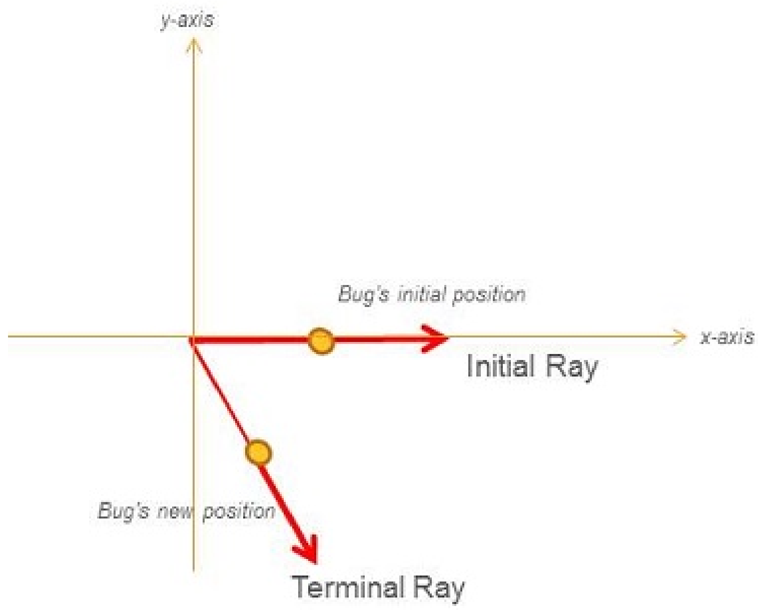
Sometimes we need to locate a point on one ray of an angle with greater precision than we used in the diagram above. When this is required, we imagine a rectangular grid is positioned so that the following conditions are met.

- The vertex of the angle lies at the origin of the grid.
- The initial ray of the angle lies on the positive half of the horzintal axis.

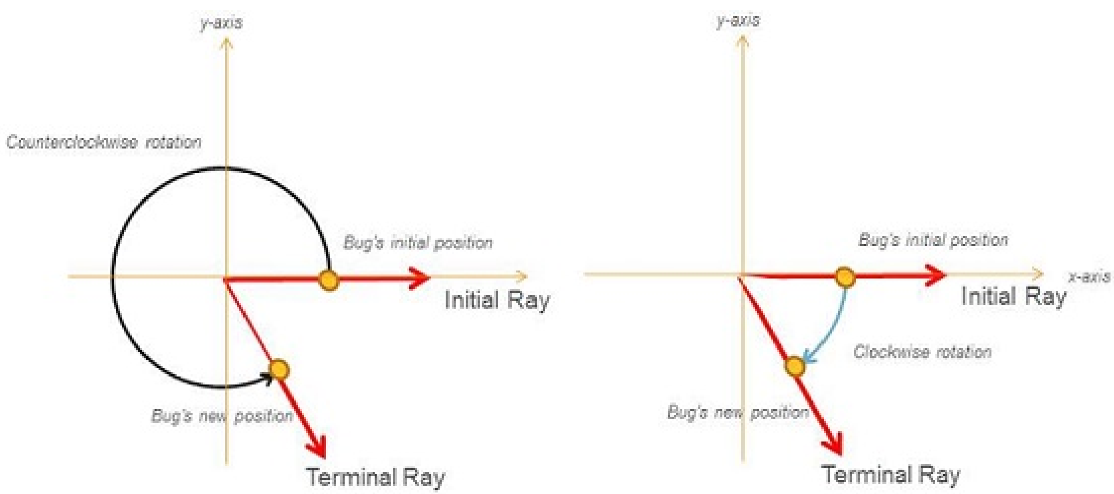
When these conditions are met, we say that the angle is in *standard position* on the grid. When an angle is in standard position on a grid, we have a systemaitc way of identifying points on both rays defining the angle. The diagram below shows the bug's rotation angle placed in standard position.



Here is the previous diagram showing the angle in standard position made by the bug as it rotates from its initial position to its new position, only this time with no direction arc to indicate which direction the bug traveled.



There are two ways the bug could have gotten from its initial position to its new position by traveling on the tip of the fan blade — the blade could have rotated in the *counterclockwise* direction, or it could have rotated in the *clockwise direction*. These two possibilities are illustrated in the figure below.



Compare the arcs cut by this angle from the two-foot circle when we rotate in the counterclockwise direction versus the clockwise direction. It is clear from the diagrams that these arcs have different lengths, and this means these arcs give rise to different radian measures for each of the two rotation angles.

In order to tell the difference between the clockwise-oriented radian measure and the counterclockwise-oriented radian measure for a rotation angle, we adopt the following convention:

Distinguishing Rotation Direction

Radian measure will be positive for counterclockwise rotation and negative for clockwise rotation.

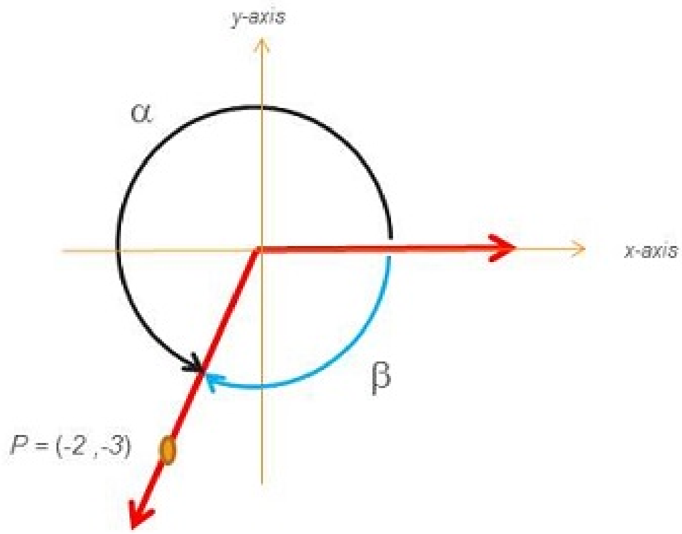
In other words, when you are told that an angle has radian measure -3.17 `rad`, this means that the measure represents a rotation through 3.17 radius lengths *in the clockwise direction* along the circumference of any circle centered on the angle vertex.

Problem 4. As the fan blade rotates, it traces out a circle with a two-foot radius. Suppose the bug rotates two-thirds of the way around this circle in the clockwise direction. What is the clockwise-oriented radian measure for this rotation angle?

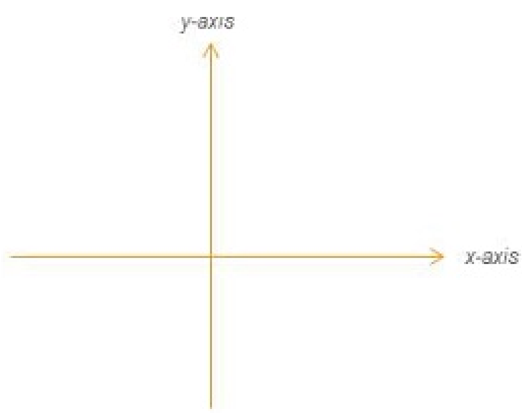
Problem 5. If the bug rotates two thirds of the way around the circle in the clockwise direction, how far around the circle in the *counterclockwise* direction would the bug have to rotate to form the same angle? What is the *counterclockwise*-oriented radian measure for this rotation angle?

Problem 6. Suppose the bug rotates through an angle having radian measure $-\dfrac{5\pi}{8}$ `rad`. What is the *counterclockwise*-oriented radian measure for this angle?

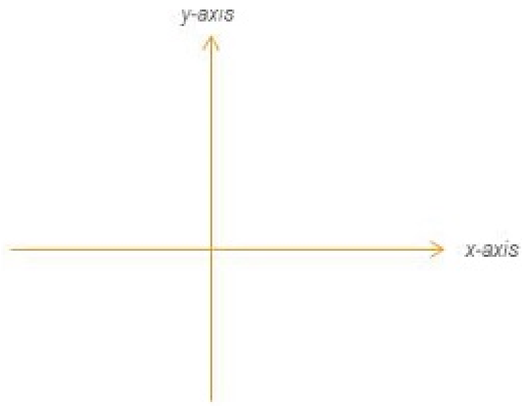
Problem 7. Consider the two rotation angles shown in the diagram below, one with counterclockwise-oriented radian measure α and the other with clockwise-oriented radian measure β. Explain why we have $\alpha - \beta = 2\pi$ `rad`. (Remember, β is negative.)

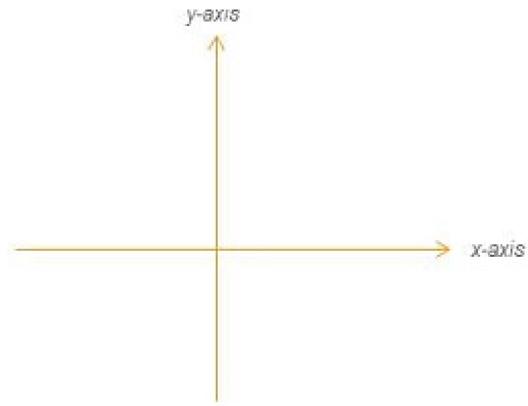


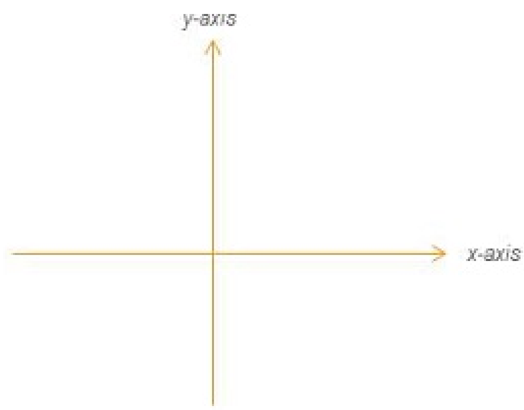
Problem 8. Let's return to the fan problem again. We know that an angle measure of 2π `rad` represents represents one complete rotation around the two-foot circle in the counterclockwise direction (so the bug's initial position and new position are the same). Suppose you are told that the bug has rotated through an angle having radian measure 3π `rad`. How would you explain the bug's motion around the fan? Draw this angle and its direction arc on the grid below.



Problem 9. On the first grid below, draw a rotation angle in standard position that has radian measure $\frac{\pi}{4}$ `rad`. On the second grid, draw a rotation angle in standard position having radian measure $-\frac{7\pi}{4}$ `rad`, and on the third grid, draw a rotation angle in standard position that has radian measure $\frac{9\pi}{4}$ `rad`. Do you notice anything about these three angles?







Problem 10. Suppose that someone points out to you that

$$\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi \qquad -\frac{7\pi}{4} = \frac{\pi}{4} - 2\pi$$

How could you use this to explain what you noticed in Problem 9? Think in terms of rotations.

Two rotation angles are said to be *coterminal* if they share the same initial ray and terminal ray. Coterminal angles represent different ways to get *from* one fixed point on a circle *to* another fixed point on the same circle. The two rotation angles shown in Problem 7 above are coterminal. The three rotation angles you drew in Problem 9 are also coterminal.

There are infinitely many rotation angles that are coterminal with any given rotation angle.

Coterminal rotation angles all differ by a series of complete rotations. Consequently, all coterminal angles will have radian measures that differ by integer multiples of 2π.

Problem 10. Suppose the measure for an angle is $\theta = 3.50$ `rad`. The value $\alpha = 22.35$ `rad` is the radian measure of another angle coterminal with the one have measure θ.. By how many complete rotations (and in what direction) does the measure α differ from the measure θ?

Problem 11. Suppose the measure for an angle is $\theta = -\frac{\pi}{5}$ `rad`.

Part (a). What is the radian measure β of the angle formed by adding three clockwise rotations to the angle with with measure θ?

Part (b). What is the radian measure α of the angle formed by adding five counterclockwise rotations to the angle with measure θ?

Supplemental Problems

1. Suppose the angle defined by the initial position and new position of the bug has degree measure $38°$. What is the length of the arc the bug traveled through?
2. Suppose the bug travels through an arc of five feet as it moves from its initial position to its new position counterclockwise in the two-foot radius fan. What is the radian measure of the angle defined by these positions?
3. Suppose the angle defined by the bug's initial position and its new position has radian measure 3.17 `rad`. What is the length of the arc the bug traveled through? Assume the fan has a two-foot radius.
4. Jade turns up the speed of the fan so that its blades make one complete rotation every second. What is the angular speed for the fan?
5. Suppose the angular speed of the fan is $\omega = 1.17$ `rad` per second. What is the length of the arc the bug travels through in two seconds? Assume the fan has a two-foot radius.

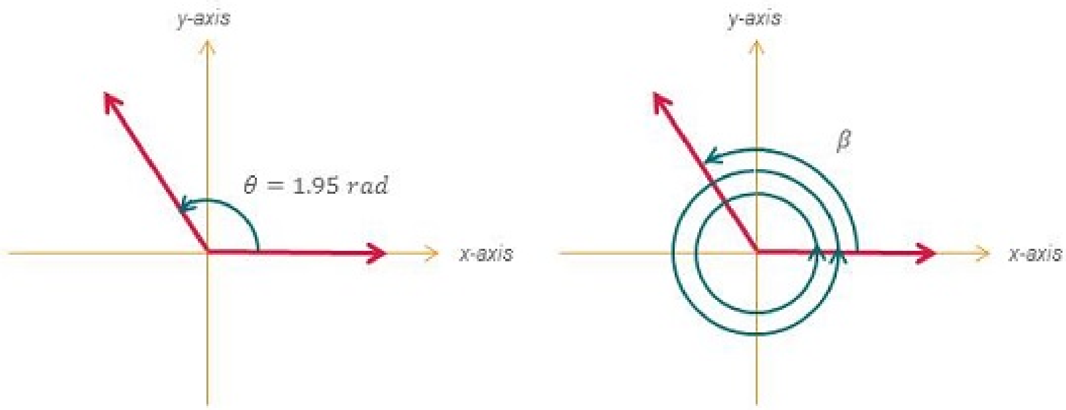
6. What would be a *clockwise-oriented* measure for angles coterminal to those whose radian measure is given below?

(a) $\theta = \dfrac{4\pi}{3}$ rad (b) $\theta = \dfrac{8\pi}{5}$ rad (c) $\theta = 2.45$ rad

7. What would be a *counterclockwise-oriented* measure for angles coterminal to those whose radian measure is given below?

(a) $\theta = -\dfrac{4\pi}{3}$ rad (b) $\theta = -\dfrac{\pi}{2}$ rad (c) $\theta = -1.73$ rad

8. The diagram below shows the radian measures for two coterminal rotation angles. If $\theta = 1.95$ rad, what is the value of β? (The value of β must measure the complete rotations shown.)

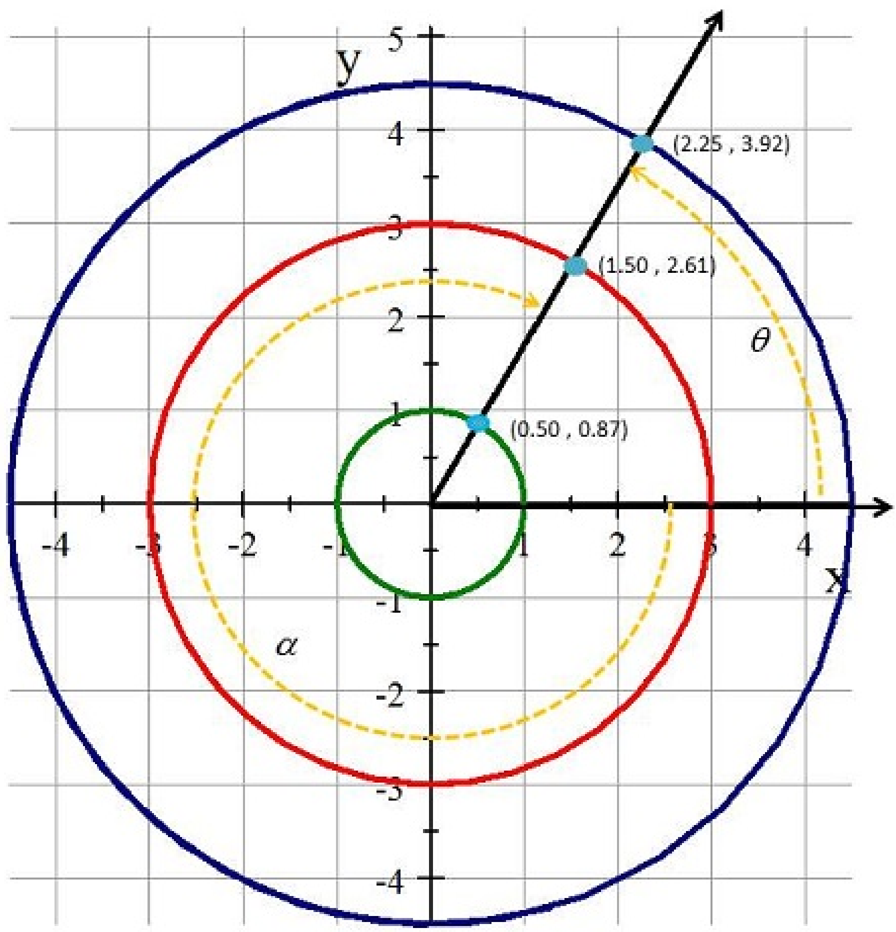


9. Suppose a rotation angle has measure $\theta = 1.88$ rad. An angle coterminal to this one is obtained by adding seven complete clockwise-oriented rotations to the original angle. What is the radian measure β of this new angle?

3 Investigation 3 — The Trigonometric Functions

In the last two investigations, we introduced radian measure for angles and explored some of the notation and conventions we use when reasoning with rotation angles. In this investigation, we will introduce three very important functions associated with the measure of angles.

The diagram below shows two coterminal rotation angles in standard position, one with radian measure θ, and the other with radian measure α. It also shows the points where the terminal side of this angle intersects three different circles centered at the origin.



1. The *tangent function* takes as input the radian measure of an angle in standard position and produces the slope of its terminal ray as output. If u is the radian measure of the angle, and m is the slope of its terminal ray, then we let $m = \tan(u)$ represent this relationship. What is the value of $\tan(\theta)$ in the diagram above? What is the value of $\tan(\alpha)$ in the diagram above?

2. How would the values of $\tan(\theta)$ and $\tan(\theta + 2\pi)$ compare? Explain the reasoning behind your answer.

3. Consider the value of the x-coordinate for each of the three points shown above. In each case, what *percentage of the radius* of the circle is this value (written as a decimal) ? Your answer will only be accurate to the nearest hundredth. What do you notice?

4. Consider the value of the y-coordinate for each of the three points shown above. In each case, what *percentage of the radius* of the circle is this value (written as a decimal) ? (Your answer will only be accurate to the nearest hundredth.) What do you notice?

Suppose that an angle in standard position has radian measure u and let $P = (x, y)$ be the point where the terminal ray of this angle intersects the circle of radius R centered at the origin. The *cosine function* takes the radian measure u as its input and produces as its output *the percentage of* R left or right of the y-axis we must move to reach the x-coordinate of P. (The precentage is negative if we have to move left and positive if we have to move right.) Written as a decimal, this precentage is simply the value of x divided by the value of R.

We use the special symbol "cos" as the name of this function. In symbols, we have

$$\cos(u) = \frac{x}{R}$$

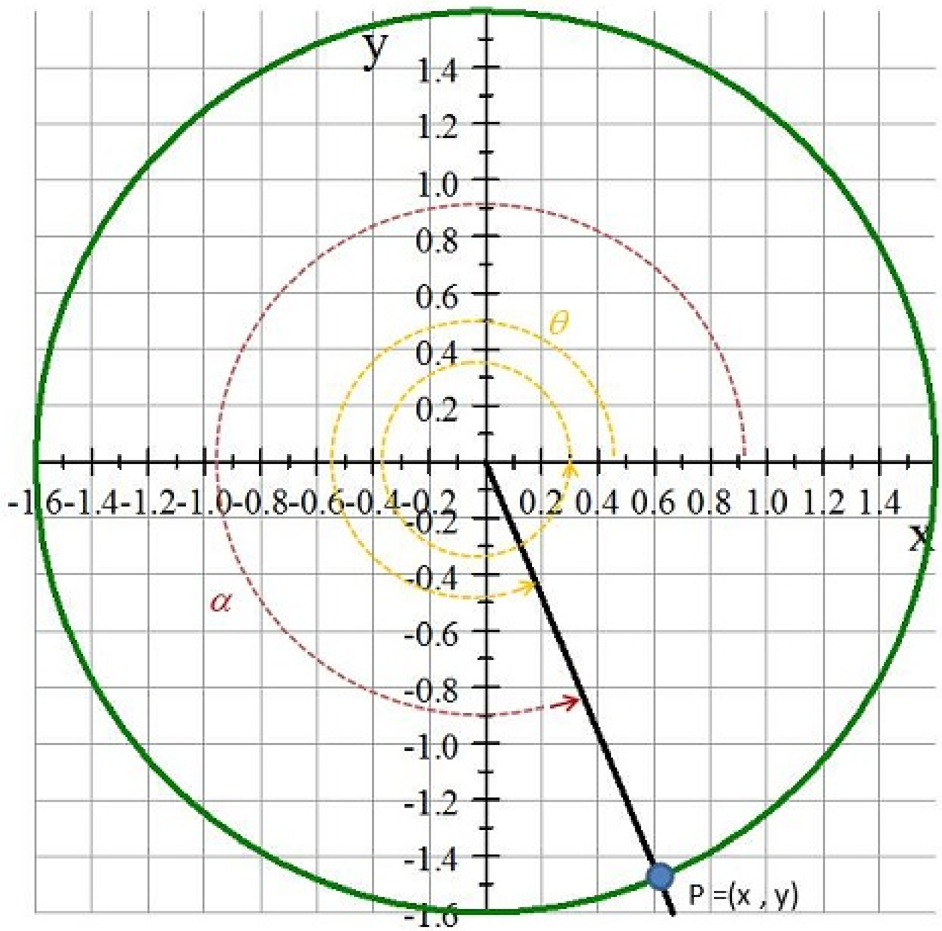
The *sine function* takes the radian measure u as its input and produces as its output *the percentage of* R above or below the x-axis we must move to reach the y-coordinate of P. (The precentage is negative if we have to move below and positive if we have to move above.) Written as a decimal, this precentage is simply the value of y divided by the value of R.

We use the special symbol "sin" as the name of this function. In symbols, we have

$$\sin(u) = \frac{y}{R}$$

5. In the diagram above, what is the value of $\sin(\theta)$ and $\sin(\alpha)$? What is the value of $\cos(\theta)$ and $\cos(\alpha)$?

6. The diagram below shows two coterminal angles in standard position with radian measure θ and α.



Part (a) Estimate the coordinates of the point P.

Part (b) Use these estimates to approximate the value of $\sin(\theta)$ and $\cos(\theta)$ and the values of $\sin(\alpha)$ and $\cos(\alpha)$.

Part (c) What is the slope of the terminal ray for these coterminal angles? What is the approximate value of $\tan(\theta)$ and $\tan(\alpha)$?

7. Suppose two coterminal angles in standard position have radian measures u and v, respectively. How would $\sin(u)$ and $\sin(v)$ compare? How would $\cos(u)$ and $\cos(v)$ compare? Explain your reasoning.

8. Mavis starts skiing around a circluar trail whose radius is four kilometers. Assume the center of the track lies at the origin of a grid, and suppose her starting point is on the positive x-axis. (This means her starting point is $(4, 0)$.) She decides to stop and take a break at the point $(-2.68, 2.97)$. What percentage of the radius of the track has she moved left or right of the y-axis? What percentage of the radius has she moved above or below the x-axis?

9. Imagine a ray extending from the center of the track to the point where Mavis is resting. This serves as the terminal ray for an angle in standard position. If u is any radian measure for this angle, what is the value of $\cos(u)$? What is the value of $\sin(u)$? What is the value of $\tan(u)$?

10. Suppose that $P = (x, y)$ is a point on a circle of radius R centered at the origin. Explain why it is not possible for the value of x or the value of y to be more than 100% of the value of R.

11. Consider a circle of radius R centered at the origin. Find a point P on the circle whose x-coordinate is 100% of R right of the y-axis. Find a point Q on the circle whose y-coordinate is 100% of R left of the x-axis.

Supplemental Problems.

1. Suppose that $P = (8, -3)$ lies on the terminal ray of an angle in standard position. If θ is the radian measure of this angle, determine the values of $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$.

2. Suppose that $P = (-2, 4)$ lies on the terminal ray of an angle in standard position. If θ is the radian measure of this angle, determine the values of $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$.

3. Explain why the following statement is true — "If α and β are radian measures for two coterminal angles in standard position, then $\sin(\alpha) = \sin(\beta)$ and $\cos(\alpha) = \cos(\beta)$.

4. If α and β are radian measures for two coterminal angles in standard position, then is it true that $\tan(\alpha) = \tan(\beta)$? Think carefully before answering.

5. Explain why the following statement is true — "If θ is the radian measure for an angle in standard position, then $\sin(\theta) = \sin(\theta \pm 2\pi)$ and $\cos(\theta) = \cos(\theta \pm 2\pi)$.

6. The *reciprocals* of the three trigonometric functions appear often enough in mathematical formulas that they are also given special names. Let θ be the radian measure of an angle in standard position.

   Secant of θ is defined by $\sec(\theta) = \dfrac{1}{\cos(\theta)}$ *Cosecant* of θ is defined by $\csc(\theta) = \dfrac{1}{\sin(\theta)}$

   Cotangent of θ is defined by $\cot(\theta) = \dfrac{1}{\tan(\theta)}$

   (a) Suppose that $P = (4, -3)$ lies on the terminal ray of an angle in standard position. If θ is the radian measure of this angle, determine the values of $\sec(\theta)$, $\csc(\theta)$, and $\cot(\theta)$.

   (b) Let θ be the radian measure of an angle in standard position. Suppose we know that this angle cuts an arc exactly half the circumference of a circle having a four foot radius. What is the value of $\sec(\theta)$, $\csc(\theta)$, and $\cot(\theta)$?

7. Draw a circle whose center lies at the origin of a rectangular grid. You may choose the radius to be whatever you like. Let x represent values on the horizontal axis and let y represent values on the vertical axis of this grid.

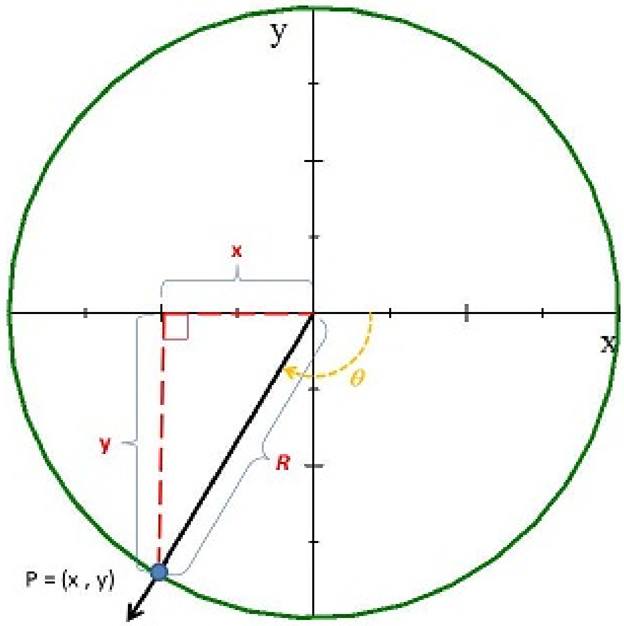
   (a) Plot the four points where your circle intersects the x and y axes along with their coordinates.
   (b) Suppose a, b, c, d are the radian measures of the angles in standard position whose terminal rays lie on the x and y axes. What are the radian measures of these angles?
   (c) Which of these angle measures produces an output of 0 from the cosine function?
   (d) Which of these angle measures produces an output of 0 from the sine function?
   (e) Are there any other angle measures that will produce an output of 0 for the sine or cosine function? Think carefully and explain your reasoning.

8. Use your answers to Problem 7 to determine *all* angle measures for which the secant function is undefined.

9. Use your answers to Problem 7 to determine *all* angle measures for which the cosecant function is undefined.

4 Investigation 4 — Relationships Between the Trigonometric Functions

In the last investigation, we introduced the sine, cosine, and tangent functions. In this investigation, we will introduce a few of the way these functions relate to one another. The diagram below shows an angle in standard position with radian measure θ along with a circle of radius R centered at the origin. Consider the point $P = (x, y)$ where the terminal side of the angle intersects the circle.



If we drop a vertical line segment down from the point P, this line intersects the x-axis at a right angle. The length of this segment is $|y|$, while the length of the horizontal line segment between the origin and this intersection point has length $|x|$. These segments form the legs of a right triangle whose hypotenuse has length R. Now, since $|y|^2 = y^2$, the Pythagorean Theorem tells us

$$R^2 = x^2 + y^2$$

1. Suppose an angle is in standard position and has radian measure θ. Suppose the point $P = (-2, 3)$ is on the terminal ray of the angle.

Part (a) What is the exact distance R between P and the origin? (Do not give a decimal approximation.)

Part (b) What are the exact values of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$?

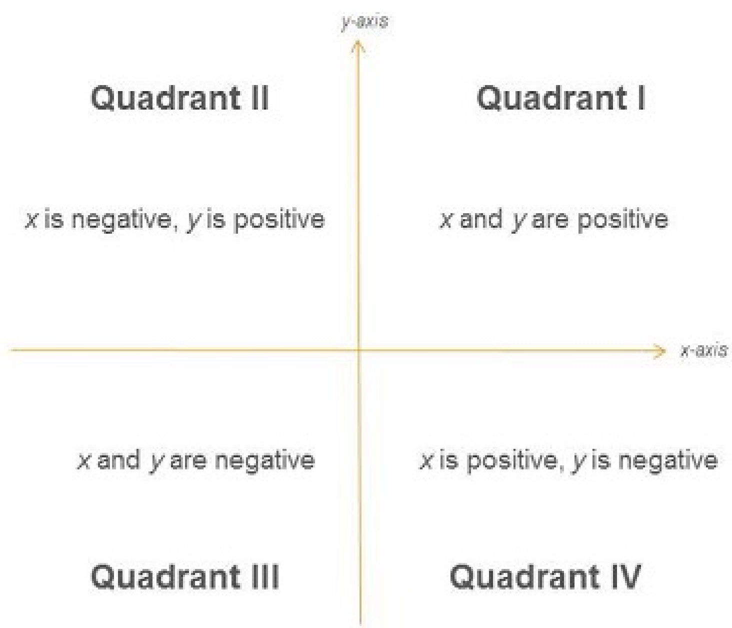
2. How can we rewrite the equation $R^2 = x^2 + y^2$ to obtain the equivalent equation $1 = [\cos(\theta)]^2 + [\sin(\theta)]^2$?

When working with trigonometric functions, it is common to write $\sin^2(\theta)$ in place of $[\sin(\theta)]^2$. It is common to do this for the cosine and tangent functions as well. Adopting this convention, we have the so-called *Pythagorean Identity* relating the outputs of the sine and cosine functions — for any angle in standard position with radian measure θ,

$$1 = \cos^2(\theta) + \sin^2(\theta)$$

Notice that the Pythagorean Identity can be rewritten as either $\cos^2(\theta) = 1 - \sin^2(\theta)$ or $\sin^2(\theta) = 1 - \cos^2(\theta)$. Consequently, if we happen to know the value of either $\sin(\theta)$ or the value of $\cos(\theta)$, we can determine the value of the other *at least up to a plus or minus sign.*

A rectangular grid naturally divides the plane into four quadrants. It is customary to name these quadrants in the counterclockise direction as shown in the diagram below.

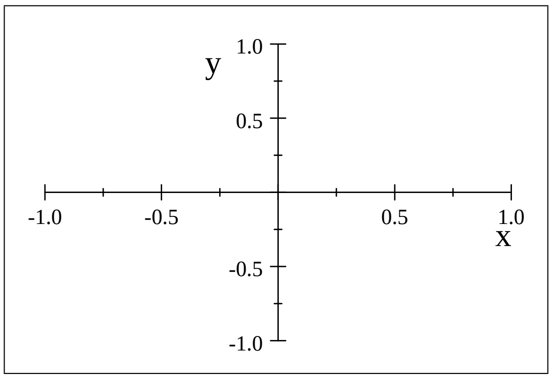


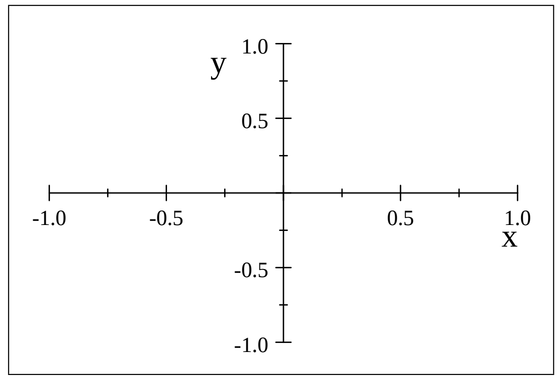
If an angle is in standard position, we say that angle *lies in a quadrant* provided it's terminal ray is in that quadrant. For example, if an angle in standard position has the point $P = (-3, 1)$ on its terminal ray, then this angle lies in Quadrant II. We know this because this is the quadrant where the first coordinate of a point is negative and the second coordinate is positive.

3. Suppose an angle in standard position lies in Quadrant III. If θ is any radian measure of this angle and $\sin(\theta) = -\dfrac{1}{4}$, use the Pythagorean Identity and the quadrant information to determine the value of $\cos(\theta)$.

4. Suppose an angle is in standard position and has radian measure θ. If the terminal ray of this angle is vertical, why is there a problem defining the tangent of θ?

5. On each of the grids below, draw a different angle in standard position whose terminal ray is vertical. What is the radian measure of each angle you made, and how do you know?





6. Suppose an angle is in standard position and suppose θ is the radian measure for this angle. As long as the terminal ray of the angle is not vertical, use the definitions of the sine and cosine functions and some algebra to explain why

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

7. Suppose that an angle in standard positive has radian measure θ. Furthermore, suppose we know the angle lies in Quadrant II and $\tan(\theta) = -\dfrac{3}{5}$.

Part (a) Millwood says that this information implies $\sin(\theta) = 3$ and $\cos(\theta) = -5$. Explain why Problem 2 makes this impossible.

Part (b) Find the correct values of $\sin(\theta)$ and $\cos(\theta)$.

8. Arnie is skiing in the clockwise direction around a circular track with a 5.5 kilometer radius. Suppose the center of the track is placed at the origin of a rectangular grid so that Arnie's starting point is on the positive x-axis. Arnie stops for a rest at a point P on the track.

Part (a) If Arnie has traveled 15% of the track radius to the left of the y-axis, what is the x-coordinate of the point P?

Part (b) What is the y-coordinate of the point P?

Part (c) Let θ be any radian measure for the angle in standard position whose terminal side passes through the point P. What is the value of $\tan(\theta)$?

Supplemental Problems.

1. Let A be an angle in standard position whose radian measure is θ. If the angle lies in Quadrant III and $\cos(\theta) = -\frac{1}{4}$, what is the value of $\sin(\theta)$ and $\tan(\theta)$?

2. Let A be an angle in standard position whose radian measure is θ. If the angle lies in Quadrant II and $\sin(\theta) = \frac{2}{3}$, what is the value of $\sec(\theta)$ and $\cot(\theta)$?

3. Let θ be the radian measure of any angle in standard position. Use the Pythagorean Identity and the definition of the secant function show that
$$\sec^2(\theta) = 1 + \tan^2(\theta)$$
as long as all expressions are defined. Hint: Divide both sides of the Pythagorean Identity by $\cos^2(\theta)$.

4. Let A be an angle in standard position whose radian measure is θ. If the angle lies in Quadrant IV and $\tan(\theta) = -\frac{5}{4}$, what is the value of $\sec(\theta)$ and $\sin(\theta)$?

5. Let A be an angle in standard position whose radian measure is θ. Explain why we always have $-1 \leq \cos(\theta) \leq 1$ and $-1 \leq \sin(\theta) \leq 1$.

6. Let A be an angle in standard position lying in Quadrant IV whose radian measure is θ, and suppose that $\tan(\theta) = -\dfrac{4}{3}$.

   (a) Explain why it is wrong to conclude that $\cos(\theta) = 3$ and $\sin(\theta) = -4$, even though $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.

   (b) Determine the correct values for $\cos(\theta)$ and $\sin(\theta)$.

7. Simplify the following expressions as much as possible. Your final form of each expression should contain only sine and cosine factors.

   (a) $\dfrac{\sin(a)}{\csc(a)} + \dfrac{\cos(a)}{\sec(a)}$ (b) $\dfrac{\tan(b)\cot(b)}{\csc(b)}$ (c) $\dfrac{\sin(\theta)}{\sec^2(\theta)} + \dfrac{1}{\csc^3(\theta)}$

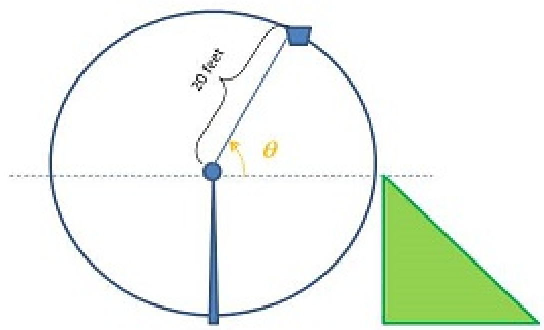
5 Investigation 5 — Sinusoids (Part 1)

A *sinusoid* is a function that has the form

$$y = f(x) = a\sin(\omega x + \theta) + v \qquad \text{or} \qquad y = g(x) = a\cos(\omega x + \theta) + v$$

where a, θ, v, and ω are all constants. In this investigation, we will begin to explore the meaning of these constants and how sinusoids are used to model certain relationships between changing quantities.

The diagram below represents a ferris wheel that has a twenty foot radius. The passengers board the wheel from a platform as shown.



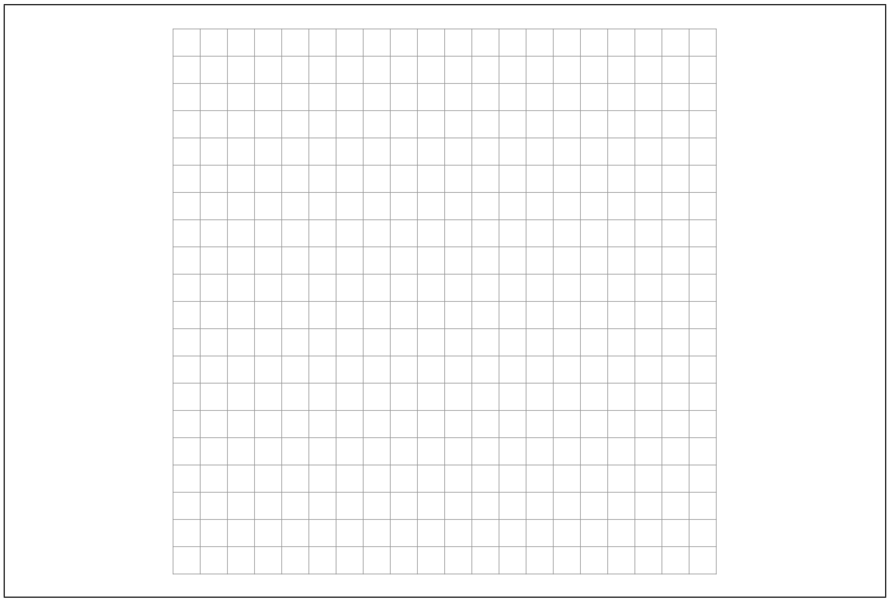
Loomis boards the ferris wheel. The ferris wheel starts to rotate counterclockwise, making one complete rotation every three minutes. As shown in the figure above, let θ be the radian measure of the counterclockwise-oriented angle in standard position formed by the horizontal line passing through the center of the wheel and the metal beam supporting Loomis's gondola.. Let y represent Loomis' vertical distance in feet *above* the platform. (Let negative values of y denote vertical distance below the platform.)

1. Recall the definition of angular speed from Problem 3 of Investigation 2. What is Loomis' angular speed?

2. Fill in the table below.

Number of Rotations	Value of θ	Value of y	Value of $\sin(\theta)$
0			
1/4			
1/2			
3/4			
1			
5/4			
3/2			
7/4			
2			

3. Let f be the function that gives the value of y as a function of the value of θ. Use the information in your table to construct a sketch of the graph of f on the grid below. Be sure to label your axes.



4. Construct the formula for $y = f(\theta)$. Explain how you arrived at your answer.

5. Complete the following table relating the value of θ to the number of minutes that have passed since the ferris wheel started rotating.

Number of Minutes Passed	Value of θ
0	
0.75	
1.50	
2.25	
3.00	

6. Let t be the number of minutes that have passed since the ferris wheel started rotating and let g be the function that gives the value of θ in terms of the value of t. Use your table to construct a formula for the function g.

7. What is the relationship between Loomis' angular speed and the formula you created in Problem 6?

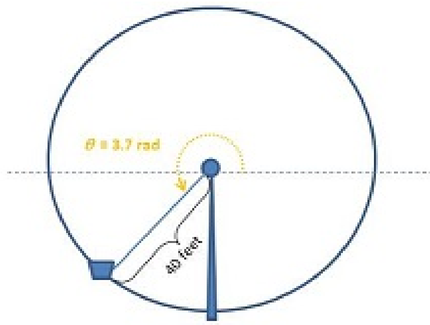
8. Construct the formula for the composite function $f \circ g$. What is the input quantity for this function? What is the output quantity?

9. When Loomis is at the lowest point on the ferris wheel, he is three feet above the ground. Let h represent Loomis' height in feet *above the ground* at any position on the ferris wheel. How could you construct the formula for a function $h = k(t)$ that gives Loomis' height above the ground in terms of the number of minutes that have passed since the ferris wheel started rotating?

10. Imagine a vertical line drawn through the center of the ferris wheel. Construct a formula for the function $x = j(t)$ that gives Loomis' horizontal distance x in feet *to the right* of this line as a function of the number t of minutes that have passed since the ferris wheel started rotating.

11. When Loomis is at the leftmost point on the ferris wheel, he is ten feet to the right of a concession stand located on the ground below. Let d represent Loomis' horizontal distance to the right of this concession stand at any position on the ferris wheel. Construct a formula $d = m(t)$ that gives the value of d in terms of the number t of minutes that have passed since the ferris wheel started rotating.

12. Suppose that Maya boards a different ferris wheel having a forty-foot radius. Maya boards the ferris wheel at the point shown below. Once Maya is onboard, the ferris wheel starts rotating counterclockwise; and Maya makes one complete rotation every half minute.



Part (a) What is Maya's angular speed?

Part (b) As Maya rotates around the wheel, let α be the radian measure of the counterclockwise-oriented angle in standard position formed by the horizontal line passing through the center of the wheel and the metal beam supporting Maya's gondola. Let t represent the number of minutes since Maya boarded the ferris wheel. Construct the linear function $\alpha = p(t)$ that gives α in terms of t. (Hint — what is α when $t = 0$?)

Part (c) Let y represent Maya's vertical distance above the horizontal line. Construct the formula for the function $y = f(t)$ that gives y in terms of t.

13. Suppose a ferris wheel starts rotating when Gerrard gets on board. Suppose further that the function

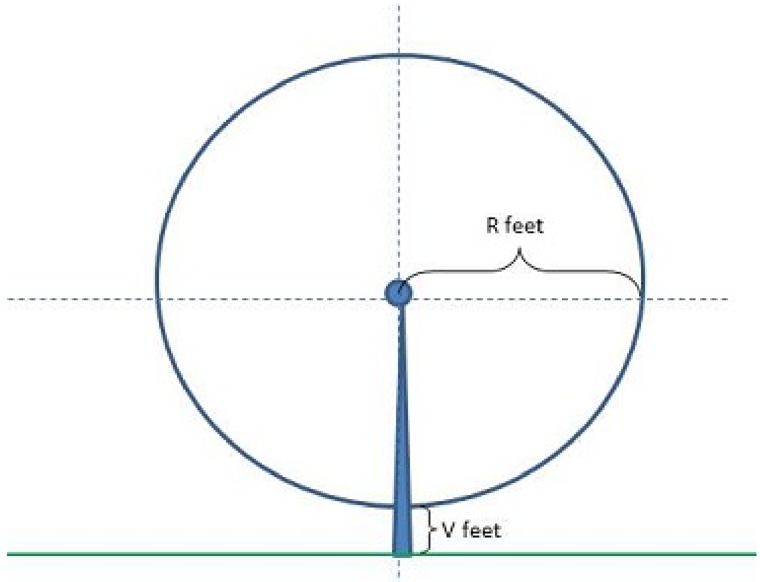
$$h = f(t) = 65 + 59 \sin\left(\frac{4}{5}t + 3\right)$$

gives Gerrard's height h in feet above the ground in terms of the number of seconds t since he boarded.

Part (a) Draw Gerrard's approximate boarding position on the diagram below.

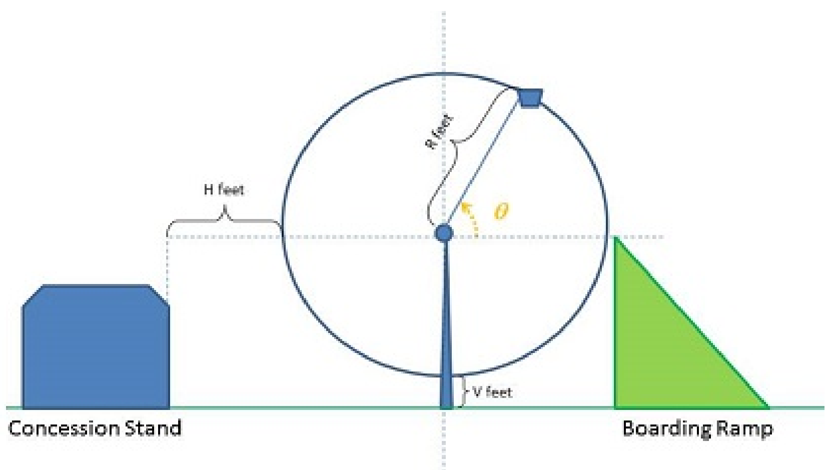
Part (b) What is Gerrard's angular speed as he rotates around the wheel?

Part (c) Determine the value of each constant in the diagram.



Supplemental Problems.

Suppose that Rochelle boards a ferris wheel from a platform as shown in the diagram below. Problems 1, 2, and 3 all refer to this diagram.



1. The ferris wheel starts rotating counterclockwise. What is the Rochelle's angular speed of the if we know

   (a) she completes one full rotation every minute?

   (b) she travels through $\frac{\pi}{6}$ radians every half minute?

(c) she travels through 50° every three-quarters of a minute?

2. Suppose Rochelle is V feet from the ground at her lowest point on the ferris wheel. Let t represent the number of minutes since the ferris wheel started rotating and let d represent Rochelle's vertical distance in feet above the boarding platform at any point on the wheel. (Let negative values of d represent distances *below* the platform.) Let h represent Rochelle's height in feet above the ground at any point on the wheel. Construct the formula for a function f that gives d in terms of t and the formula for a function g that gives h in terms of t if we know

   (a) Rochelle completes one full rotation every minute, the radius of the ferris wheel is twenty feet, and $V = 3$ feet.

   (b) Rochelle travels through $\frac{\pi}{6}$ radians every half minute, the radius of the ferris wheel is ten feet, and $V = 1.5$ feet.

   (c) Rochelle travels through 50° every three-quarters of a minute, the radius of the ferris wheel is thirty feet, and $V = 11.8$ feet.

3. Suppose Rochelle is H feet to the right of the concession stand at her leftmost point on the ferris wheel. Let t represent the number of minutes since the ferris wheel started rotating and let d represent Rochelle's horizontal distance in feet to the right of the dashed vertical line through the center of the wheel at any point on the wheel. (Let negative values of d represent distances *left* of this line.) Let h represent Rochelle's distance in feet to the right of the concession stand at any point on the wheel. Construct the formula for a function f that gives d in terms of t and the formula for a function g that gives h in terms of t if we know

   (a) Rochelle completes one full rotation every minute, the radius of the ferris wheel is twelve feet, and $H = 5$ feet.

   (b) Rochelle travels through $\frac{\pi}{6}$ radians every half minute, the radius of the ferris wheel is fourteen feet, and $H = 20$ feet.

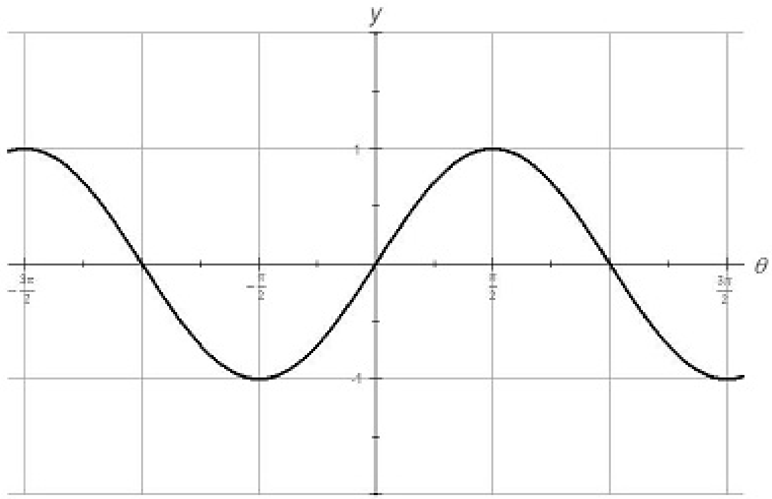
(c) Rochelle travels through 50° every three-quarters of a minute, the radius of the ferris wheel is seventeen feet, and $H = 5.75$ feet.

4. Benny boards a ferris wheel, and the wheel starts rotating counterclockwise once he is onboard. If we let t represent the number of minutes since Benny boarded the ferris wheel, what specific information about the ferris wheel, Benny's angular speed, and Benny's boarding position does the following function give us?

$$f(t) = 29.8 + 25\sin(3.9t + 5.67)$$

6 Investigation 6 — Inverse Trigonometric Functions

There are times when we need to solve for values of the input variable for trigonometric functions. Like the exponential and logarithmic functions, the trigonometric functions are not defined using algebra, so it is not possible to use algebra steps to reverse them. *Unlike* the exponential and logarithmic functions, however, the graphs of the trigonometric functions fail the horizontal line test. This means the sine, cosine, and tangent functions *do not have inverses.*

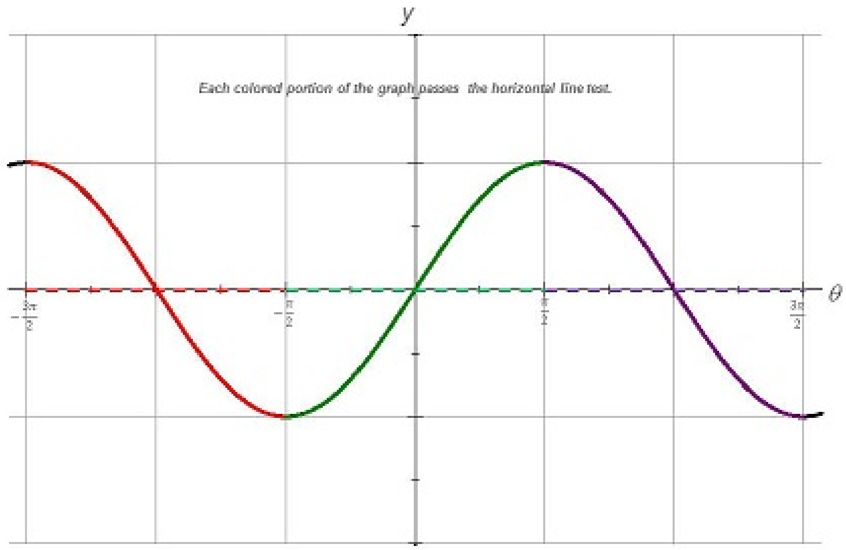


The diagram above shows a portion of the graph of the basic sine function $y = f(\theta) = \sin(\theta)$. As you can see, this function fails the horizontal line test; there are multiple input values associated with each output value. (For example, in the diagram we see the output value $y = 1$ is paired with input values $\theta = -3\pi/2$ and $\theta = \pi/2$.) This means that the process which reverses the sine function does not produce a function. There is no inverse function for the sine function.

We get around this problem by *restricting the domain* of the sine function to a set of input values where the graph does pass the horizontal line test. There are many ways we could do this. However, it is customary to adopt the following strategy when restricting the domain.

- We choose an interval where the output range of the sine function is as large as possible.
- We choose the interval to include as many "practical" angle measures as possible.

The range of the sine function is limited to the interval $-1 \leq y \leq 1$. We could restrict the domain to many different input intervals where the graph covers this range and passes the horizontal line test. Here are the domain restrictions we could make just in the diagram shown.



Of all the possible domain restrictions we could make, we choose to make the restriction $-\pi/2 \leq \theta \leq \pi/2$. This restriction gives us the green portion of the graph shown above. There are many reasons this particular restriction is made, most of which become apparent in calculus. However, there is one simple reason — this restriction includes all of the *acute* angle measures (those angles that can appear in right triangles). This fact makes this particular restriction convenient in applied trigonometry.

THE PRINCIPAL INVERSE SINE FUNCTION

The green portion of the basic sine graph shown above passes the horizontal line test and therefore represents an invertible function. The inverse of this function is called the *principal inverse sine* function or the *arc-sine function.* The principal inverse sine function is traditionally represented by

$$\theta = g(y) = \arcsin(y) \qquad \text{or} \qquad \theta = g(y) = \text{ Sin}^{-1}(y)$$

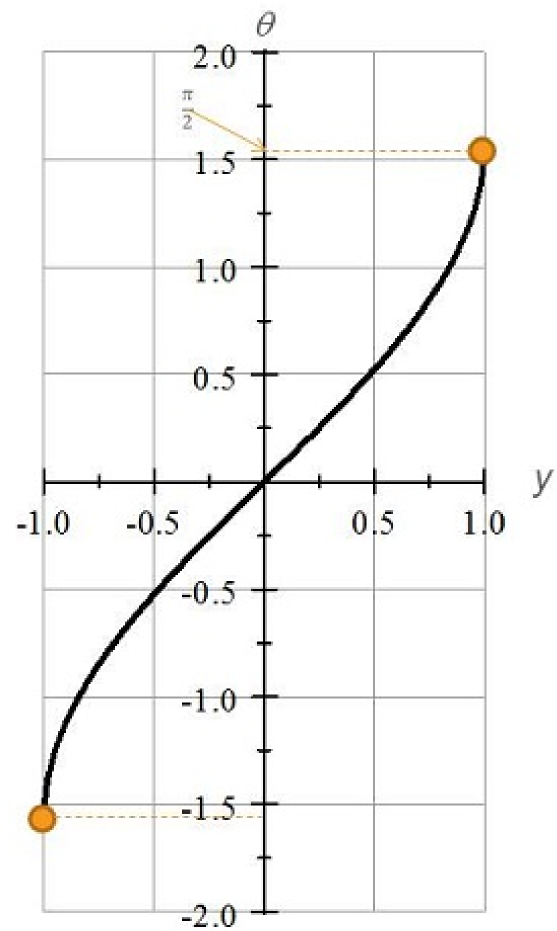
The function g is not a true inverse for the sine function, because the sine function does not have an inverse. The function g serves as a *partial inverse* for the sine function — it only reverses the sine function on the restricted input interval $-\pi/2 \leq \theta \leq \pi/2$.

1. Jerica is skiing around a circular track that has a two kilometer radius. She starts at the point $(2, 0)$ on the track, and stops at the point $(0.34, 1.97)$. Use the arc-sine function and the fact that $1.97 = 2.00\sin(\theta)$ to determine the radian measure θ of the rotation angle Jerica creates as she moves from her starting point to her stopping point. (The arc-sine function probably appears on your calculator as a $\text{Sin}^{-1}()$ key.)

2. The angle whose radian measure is $\alpha = 3.67$ `rad` and the angle whose radian measure is $\beta = -3.99$ `rad` do not lie in the restricted input interval $-\pi/2 \leq \theta \leq \pi/2$. This means that $\arcsin(\sin(\alpha)) \neq \alpha$ and $\arcsin(\sin(\beta)) \neq \beta$. Using the calculator, we know that

$$y = \sin(3.67) \approx -0.504 \qquad y = \sin(-3.99) \approx 0.750$$

The diagram below shows the graph of $\theta = g(y) = \arcsin(y)$. Use this graph to estimate the value of $\arcsin(\sin(\alpha))$ and $\arcsin(\sin(\beta))$.



The previous exercise points out that the arc-sine function is not a true inverse for the sine function. If it were a true inverse, then it would always be true that

$$\arcsin(\sin(\theta)) = \theta$$

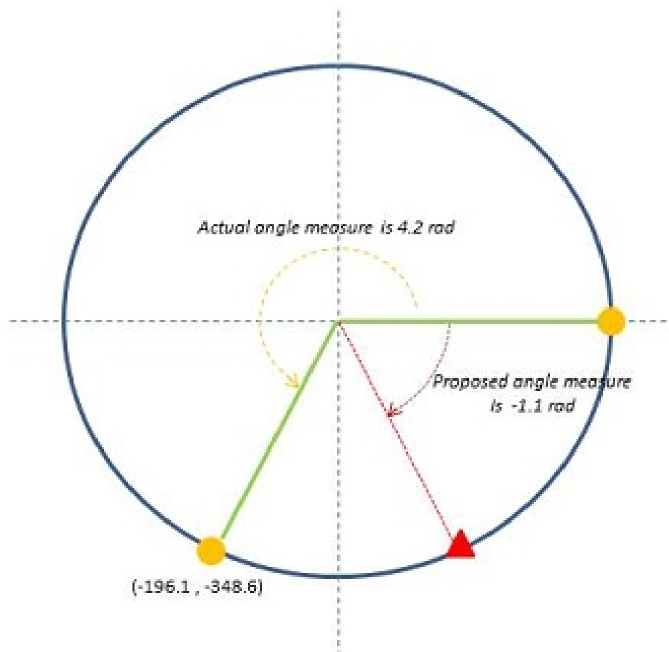
Unfortunately, the equation above holds *only* when we are dealing with the measure of a acute angle in either Quadrant I or Quadrant IV. This limitation can have some important consequences when we are solving problems where we need to determine the measure of an angle. Let's take a look at the issues this limitation can cause.

3. Reggie is running around a circular track that has a radius of 400 feet. Suppose he starts at the point $(400, 0)$ and travels through an arc in the counterclockwise direction having length 1,680 feet before he stops to rest.

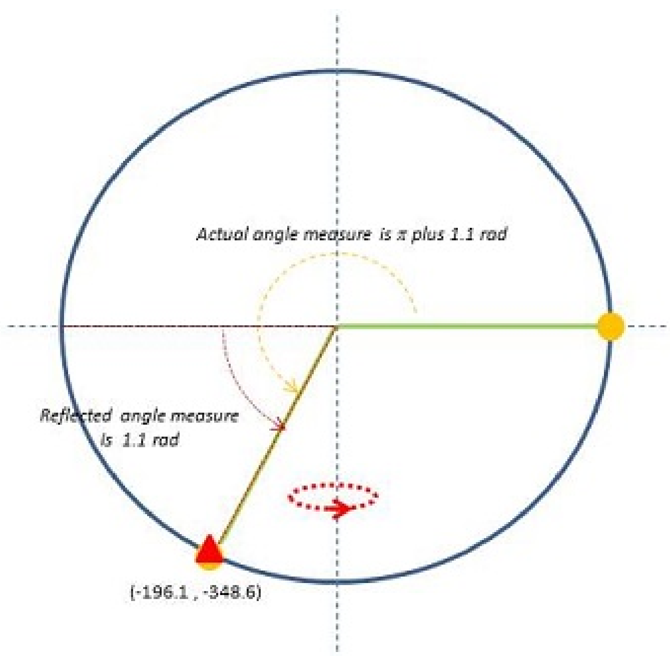
Part (a) What is the radian measure α of the rotation angle Reggie forms as he moves from his starting point to his resting point?

Part (b) Instead of knowing the arc-length Reggie traveled through, suppose we know the *coordinates* of the point where he stopped to rest —Reggie stopped to rest at the point $P = (-196.1, -348.6)$. What is the value for θ that we get if we use the arc-sine function to solve the equation $-348.6 = 400\sin(\theta)$?

We know from Part (a) that the radian measure of the rotation angle is $\alpha = 4.2$ radians, but this is not the answer we get when we solve the equation $-348.6 = 400\sin(\theta)$. What is worse, however, is that the answer we get is not even the measure of an angle *coterminal* to Reggie's true rotation angle.



There is a relationship between the proposed solution $\theta = -1.1$ `rad` and the actual angle measure $\alpha = 4.2$ `rad`, but this relationship is not clear — at least until we *reposition* the angle with measure α. Look what happens if we take the angle with measure θ and *reflect it around the vertical axis.*



We can always use the arc-sine function to solve equations of the form $b = a\sin(\theta)$ for the angle measure θ. Unfortunately, the solution we get will always be the measure of an acute angle in standard position that lies in Quadrant I or Quadrant IV. Based on the context of the particular problem we are trying to solve, *this may not be the solution we are looking for.* To find the solution that satisfies the conditions in our problem, we need extra information — in particular, we need to know what quadrant the desired angle should lie in and what the direction of rotation should be.

SOLVING EQUATIONS USING ARC-SINE

Suppose we want to solve the equation $b = a\sin(\theta)$.

- The expression $\theta = \arcsin\left(\frac{b}{a}\right)$ is always a solution, but θ will be the measure of an acute angle (in standard position) in Quadrant I or Quadrant IV.
- If we need the measure α of a *different* angle that also solves this equation, then we need more information — we need to know the quadrant of the angle and its direction of rotation.
  - If the angle we want lies in Quadrant I or Quadrant IV, then angle of measure α is coterminal with the angle of measure θ.
  - If the angle we want lies in Quadrants II or III, then the angle of measure α is coterminal with the angle of measure $\pi - \theta$.

For example, in Problem 3 Part (b), we are told that Reggie stopped at the point $P = (-196.1, -348.6)$. This point lies on the terminal ray of the angle whose radian measure we want to find. Since this point lies in Quadrant III, and since the solution to the equation $-348.6 = 400\sin(\theta)$ we obtained using the arc-sine function is $\theta \approx -1.1$ `rad`, we know that the angle whose measure we want to find will be coterminal to the angle with measure

$$\alpha = \pi - (-1.1) \approx 4.2 \texttt{ rad}$$

Since the problem also tells us that the angle is measured in the counterclockwise direction, we know α is the measure we seek.

4. Donnelle is riding a ferris wheel, and her distance above the ground in feet is given by the function

$$h = f(t) = 65 + 59\sin\left(\frac{4}{5}t - 3\right)$$

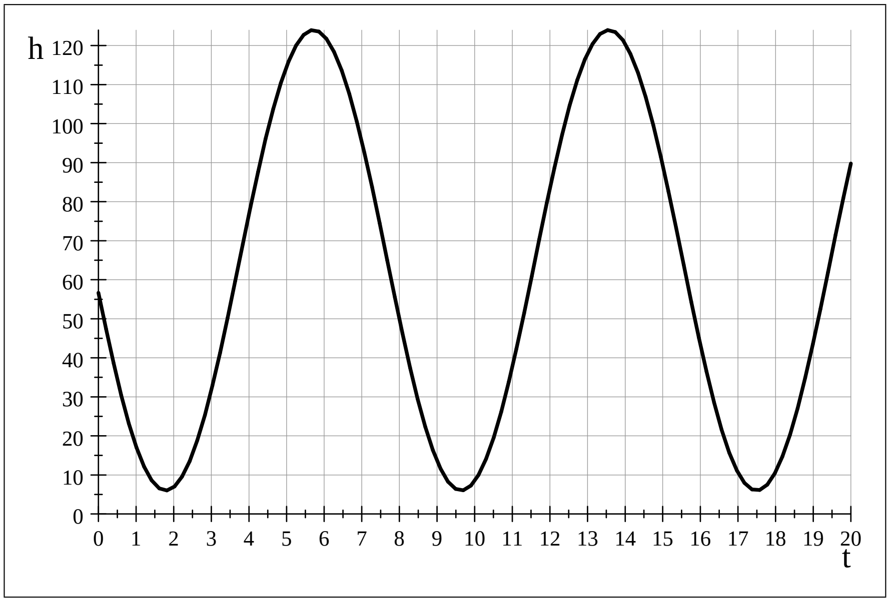
where t is the number of seconds since the ferris wheel started rotating. Use the arc-sine function to find one solution to the equation

$$45 = 65 + 59\sin\left(\frac{4}{5}t - 3\right)$$

5. The diagram below shows a graph of Donnelle's function f from the previous problem.

Part (a) On this graph, mark the approximate ordered pair (t, h) that corresponds to your solution from Problem 2.

Part (b) Use the graph to determine all of the solutions to the equation $42 = 65 + 59 \sin\left(\frac{4}{5}t - 3\right)$ that can be found from the diagram.



Part (c) There will be more solutions to this equation for values of t larger than 20 seconds. Based on the pattern you see in your solutions, what will the next two approximate solutions larger than 20 seconds be for the equation?

As Problem 5 points out, there are drawbacks to using the arc-sine function to solve equations that did not show up when we used logarithm functions to solve exponential equations, or when we used exponential functions to solve logarithmic equations. The arc-sine function can only give us *one* solution to a sinusoidal equation, even though that equation will have *infinitely many* solutions.

Example 2 *Use the properties of the arc-sine function to rewrite* $y = h(t) = \sec\left[\arcsin\left(\frac{3t}{4}\right)\right]$ *as an algebraic function of* t.

Solution. A function is *algebraic* provided its formula is constructed using only arithmetic operations. We know that the output of the arc-sine function is the radian measure of an acute angle. Let θ be the measure of this angle. This means

$$\theta = \arcsin\left(\frac{3t}{4}\right) \qquad \text{and} \qquad \sin(\theta) = \frac{3t}{4}$$

Now, if we let $P = (a, b)$ be any point on the terminal side of the angle in standard position whose measure is θ, then we know that

$$\frac{b}{\sqrt{a^2 + b^2}} = \sin(\theta) = \frac{3t}{4}$$

Therefore, we can let $b = 3t$, and we know $4 = \sqrt{a^2 + b^2}$. The second equation tells us

$$4 = \sqrt{a^2 + b^2} \Longrightarrow 16 = a^2 + 9t^2 \Longrightarrow \pm\sqrt{16 - 9t^2} = a$$

We can use the properties of the arc-sine function to say more, however. The way we have defined the arc-sine function, the angle whose measure is θ must lie in Quadrant I or Quadrant IV. In either case, *the x-coordinate of the point P must be positive.* Therefore, we know

$$h(t) = \sec\left[\arcsin\left(\frac{3t}{4}\right)\right] = \sec(\theta) = \frac{4}{\sqrt{16 - 9t^2}}$$

****

6. Use the properties of the arc-sine function to rewrite $y = h(t) = \tan\left[\arcsin\left(\frac{2}{5t}\right)\right]$ as an algebraic function.

Supplemental Problems.

1. Betula, Anne, and Marques are riding bicycles around a circular track that has a radius of 500 feet. They each start at the point $(500, 0)$ and travel counterclockwise less than one time around the track before stopping.

   (a) If Anne stops at the point $(382.42, 322.11)$, then what is the radian measure of the rotation angle she traveled through from her starting point to her stopping point?

   (b) If Marques stops at the point $(-468.22, -175.39)$, then what is the radian measure of the rotation angle he traveled through from her starting point to his stopping point?

   (c) If Betula stops at the point $(-161.64, 473.15)$, then what is the radian measure of the rotation angle she traveled through from her starting point to her stopping point?

2. Nigel decides to join the cyclists and also starts at the point $(500, 0)$. However, he rides around the track three times in the counterclockwise direction before he goes on to stop at the point $(-314.10, 389.00)$.

What is the radian measure of the rotation angle he traveled through from his starting point to his stopping point?

3. Use the arc-sine function to determine one solution to the equation $83 = 100 + 30\sin\left(\frac{2}{3}x + 5\right)$.

It is possible to define partial inverse functions for all of the trigonometric functions; however, only the partial inverse functions for the sine and tangent functions are commonly used. The *arc-tangent* function is defined in much the same way that the arc-sine function is defined — the domain of $m = f(\theta) = \tan(\theta)$ is restricted to the input interval $-\pi/2 < \theta < \pi/2$ and the inverse function is constructed on this restriction. (Note the strict inequalities — the tangent function is not defined for $\theta = \pm\pi/2$.) The function $\theta = g(m) = \arctan(m)$ gives the radian measure of an acute angle (in standard position) in Quadrant I or Quadrant IV whose terminal ray has slope m. The arctangent function has the same limitations that the arc-sine function has.

4. A ferris wheel has a radius of fifty feet, and its lowest point is ten feet off the ground. The boarding platform for this wheel is located sixty feet above ground. Wilma boards the ferris wheel, and it starts rotating counterclockwise until it malfunctions. When Wilma's gondola stops, the beam connecting it to the center of the wheel has slope $m = 2.2$. Let θ be the radian measure of the rotation angle made as Wilma moved from her starting point to her stopping point. Assume this angle is in standard position, with the boarding platform on the positive x-axis.

   (a) If Wilma was *rising* when she stopped, what quadrant is this angle in?

   (b) Use the arc-tangent function (which will appear as a Tan^{-1} key on your calculator) and the fact that $2.2 = \tan(\theta)$ to determine the radian measure of the angle θ.

   (c) Suppose Fred boarded the same ferris wheel before Wilma, and when Wilma boarded, she was *directly opposite* Fred on the wheel. When the ferris wheel stopped, what was the radian measure of Fred's rotation angle?

(d) Barney boarded the ferris wheel after Fred but before Wilma. When the ferris wheel stopped, Barney was at the point $P = (-25.24, 43.16)$. What is the slope m of the beam connecting Barney's gondola to the center of the wheel?

(e) Use the fact that $m = \tan(\theta)$ and the arc-tangent to determine the measure θ of Barney's rotation angle. Be careful about the quadrant you are in — did you find Barney's angle measure, or the angle measure for a point *directly opposite* Barney's point on the wheel?

5. Rewrite the following functions as algebraic functions of the input variable.

(a) $y = h(t) = \sec\left[\arcsin\left(\frac{2t}{3}\right)\right]$ (b) $y = f(x) = \sin\left[\arctan\left(\frac{1}{x}\right)\right]$

7 Investigation 7 — Sinusoids (Part 2)

In the last investigation, we saw that sinusoid functions can be used to model aspects of movement around a ferris wheel; and we developed some interpretation for the constants that appear in sinusoids. In this investigation, we will think more deeply about sinusoid functions and the meaning of the constants used to create them. By doing this, we will be able to consider many other relationships between changing quantities that can be modeled by sinusoid functions.

1. Ryan and Jalina each take a ball attached to some string and start twirling it vertically in front of them in the counterclockwise direction. The string attached to Ryan's ball is 1.5 feet long, while the string attached to Jalina's ball is two feet long.

Part (a) Suppose Ryan's ball has an angular speed $\omega = 4$ radians per second. How many seconds will it take for Ryan's ball to make one complete rotation?

Part (b) Suppose Jalina's ball has an angular speed $\varpi = 8$ radians per second. How many seconds will it take for Jalina's ball to make one complete rotation?

Part (c) Imagine a horizontal line passing through Ryan's hand that intersects the circle traced by the ball. Let y represent the vertical distance in feet his ball is above this line, let t be the number of seconds since he started twirling, and let θ be the radian measure of the rotation angle in between the horizontal line and the string. If Ryan started twirling the ball at $\theta = \frac{3\pi}{2}$, construct the formula for the function $y = f(t)$.

Part (d) Imagine a horizontal line passing through Jalina's hand that intersects the circle traced by the ball. Let y represent the vertical distance in feet his ball is above this line, let t be the number of seconds since she started twirling, and let θ be the radian measure of the rotation angle in standard position between the horizontal line and the string. If Jaline started twirling the ball at $\theta = \pi$, construct the formula for the function $y = g(t)$.

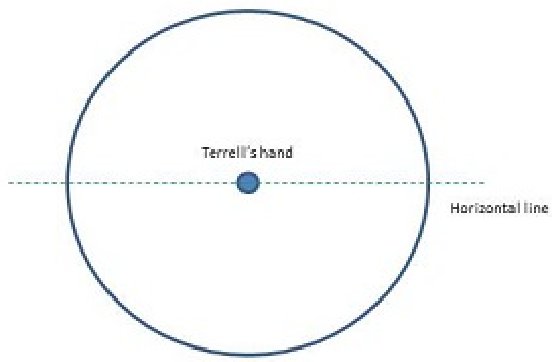
Part (e) How many seconds will it take for the outputs of the function f to start repeating and the outputs of the function g to start repeating? Explain your reasoning.

A function $y = f(x)$ is said to be *periodic* if its graph exhibits a repeating pattern. The functions f and g you created in Problem 1 are both periodic, because their outputs will repeat every time the ball makes a complete rotation. The *period* of the function is the shortest distance you must move left or right from any given point on the graph in order to reach the next point where the graph behaves exactly the same way. In Problem 1, the period of each function is just the amount of time it takes for each ball to make one complete rotation.

2. Terrell is twirling a ball in the counterclockwise direction vertically like Ryan and Jalina. The vertical distance y in feet of his ball *above the ground* with respect to the number t of seconds since he started twirling is given by the function

$$y = h(t) = 5 + 2.2 \sin\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right)$$

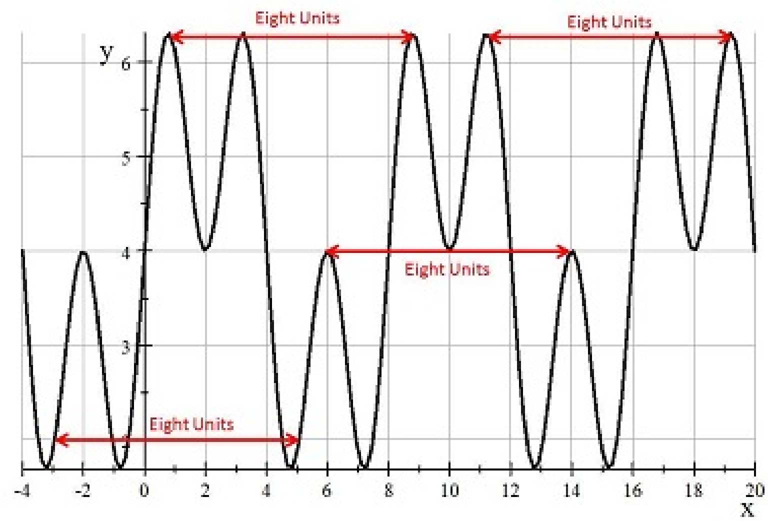
Part (a) If we let θ be the radian measure of the rotation angle in standard position between the horizontal and the string, draw the approximate starting position for Terrell's ball on the circle below.



Part (b) What is the angular speed for Terrell's ball?

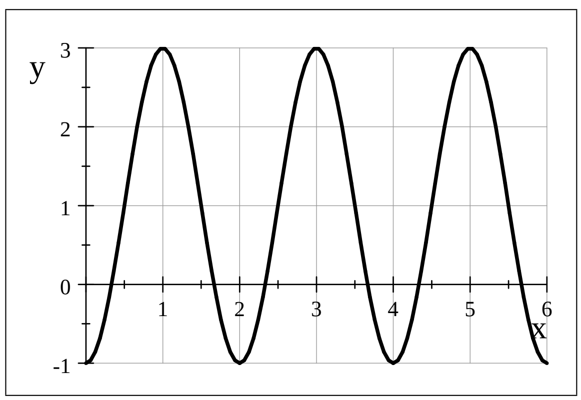
Part (c) What is the period for the function h?

The diagram below shows a function whose graph displays a repeating pattern. Since there is a regular repeating pattern to the graph, we know this function is periodic.

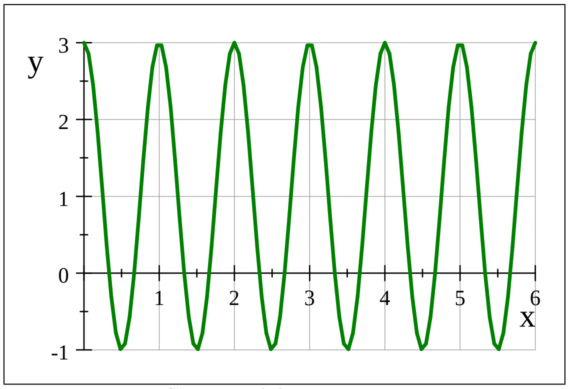


The period of this particular function is 8, because starting at any point on the graph, we must move eight units left or right before we reach a point on the graph where *exactly the same behavior* is occurring.

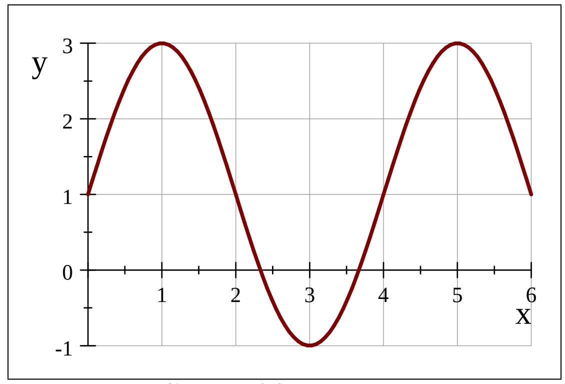
3. The diagrams below show the graph of a periodic function $y = f(x)$ and the graphs of $y = g(x) = f(2x)$ and $y = h(x) = f\left(\frac{1}{2}x\right)$.



Graph of function f



Graph of function g

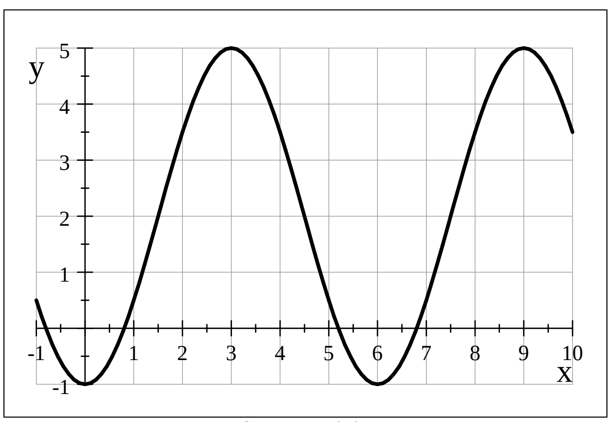


Graph of function h

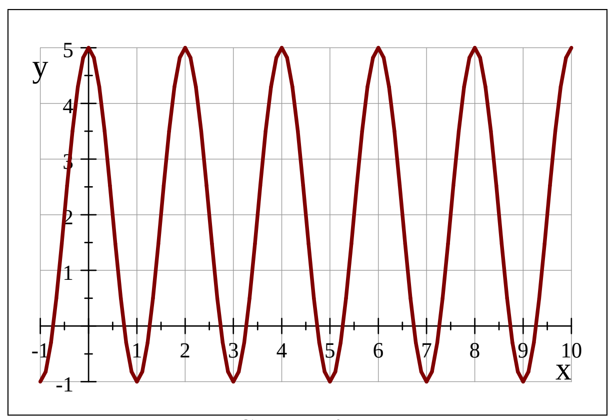
Part (a) Use the graphs to determine the period of each of the functions f, g, and h.

Part (b) How do the periods for g and h compare to the period of f?

4. The diagrams below show the graph of a periodic function $y = f(x)$ and a function $y = g(x) = f(\omega x)$. Based on the pattern you observed in Problem 3 , determine the value of the constant ω.



Graph of f



Graph of g

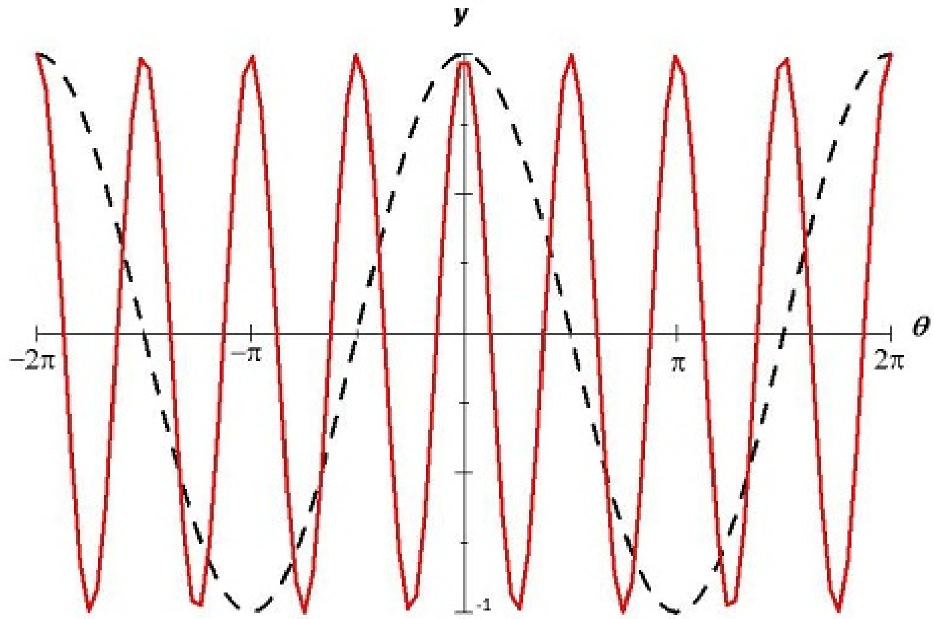
Frequency of a Periodic Function

Suppose that $y = f(x)$ is a periodic function with period p and suppose that $y = g(x) = f(\omega x)$ where ω is a constant. It is customary to call ω the *frequency* of the function g (compared to the function f). The frequency counts the number of full pattern repititions in the graph g that take place in *one* full pattern repitition of the graph f. The period q of the function g will be

$$q = \frac{p}{\omega}$$

When the function g is associated with rotating object, then the constant ω can be interpreted as the angular speed of the object, and the frequency is simply the time it takes for the object to make one complete rotation.

5. The diagram below shows the graphs of two periodic functions f and g. The dashed line is the graph of the basic cosine function.



Part (a) What is the period of the function g?

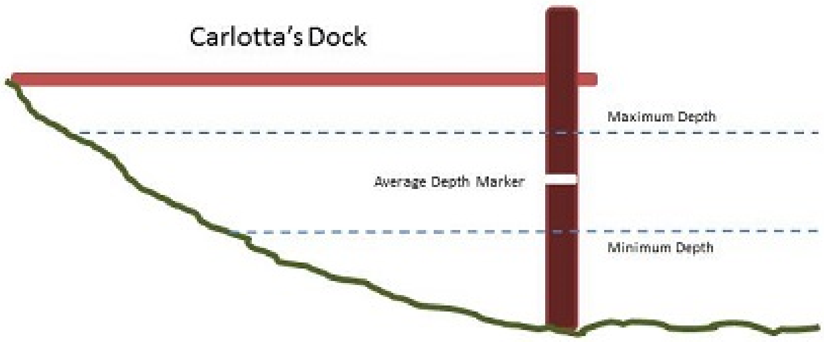
Part (b) What is the frequency of the function g compared to the basic cosine function?

We have used sinusoid functions to represents certain relationships between quantities that change as objects rotate in a circle. In particular, we have used these types of functions to represent how the vertical distance of the object above a horizontal line changes with time, and we have used these types of functions to represent how the horizontal distance of the object to the right of a vertical line changes with time.

There are many other relationships between changing quantities that can also be represented using sinusoid functions. In fact, whenever one quantity varies regularly between a fixed maximum value and a fixed minimum value with respect to another quantity, the relationship between the two quantities can be represented by a sinusoid function.

Carlotta owns a dock in a resort town located on the shore of a large bay. She needs to construct a function that gives the depth in feet of water at the end of the dock with respect to time. She knows that the depth varies from a minimum of eight feet to a maximum of twelve feet every six hours.

On one of the piers supporting the end of the dock, there is a marker at the average depth of the bay (which would be ten feet). Because it is easy for Carlotta to see this marker, she decides to start measuring the depth at a time when the water level is just *falling* past this marker. Let D represent the depth of water in feet at the end of the pier, let t represent the number of hours since Carlotta started measuring the depth of water in the bay, and let B be the function that gives the values of D in terms of the values of t.



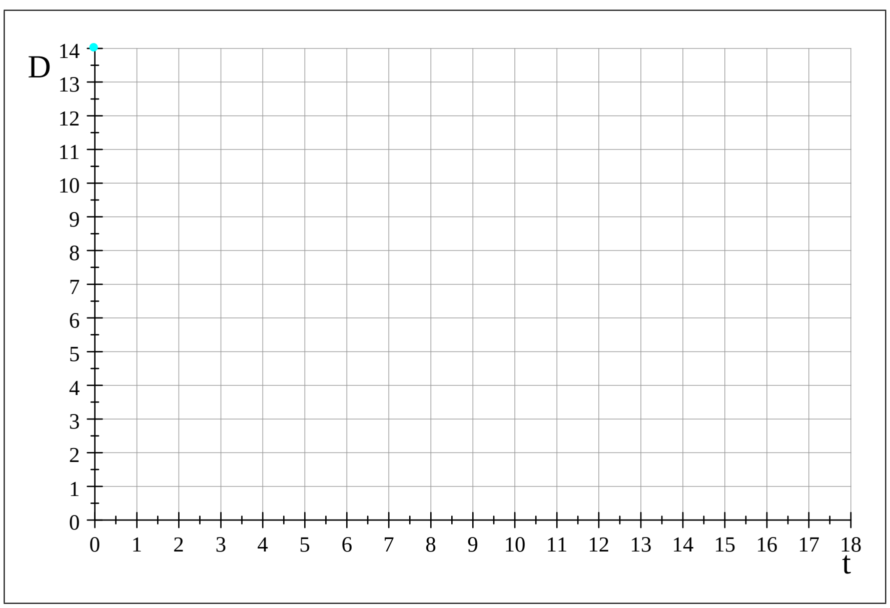
6. Carlotta knows that the function B is periodic.

Part (a) Why does Carlotta know the function is periodic?

Part (b) What is the period of the function B?

Part (c) What is the frequency ω of the function B compared to the basic sine or the basic cosine function?

7. On the grid provided, draw a sketch of what you think the graph of the function B should look like on the eighteen hour time interval shown. Be careful about how you start your graph.



The outputs of a sinusoid function will always vary between a fixed maximum value and a fixed minimum value; consequently, there is always an average value for the output of a sinusoid function. The distance between the average value of the output and the maximum (or minimum) value of the output is called the *amplitude* of the sinusoid.

8. Let's consider some of the sinusoid functions we have already constructed.

Part (a) Go back to Problem 1 and think about the function $y = f(t)$ that gives the vertical distance in feet above the horizontal for Ryan's ball. What is the average output value for f? What is the distance between the average output value and the maximum output value? Does this number appear in your formula for the function f?

Part (b) Go back to Problem 9 from Investigation 5 and think about the function $h = k(t)$ that gives Loomis' height in feet above the ground with respect to the number of minutes since the ferris wheel started rotating. What is the average output value for the function k? What is the distance between the average output value and the maximum output value? Does this number appear in your formula for the function k?

Part (c) There are four constants involved in the formula for a sinusoid function. Based on your answers to Parts (b) and (c), which of these constants represents the amplitude of the sinusoid? Do you think one of the constants represents the average output value of the sinusoid?

9. Carolotta wants to construct a sinusoid formula for her function $D = B(t)$. She needs to find constants v, a, ω, and θ so that

$$D = B(t) = v + a\sin(\omega t + \theta)$$

You determined what ω should be in Part (c) of Problem 6. Based on your work in Problem 8, what should be the values of v and a?

In order for Carlotta to complete the formula for her sinusoid, she needs to determine what the value of the constant θ should be. We developed an interpretation for the constant θ in Investigation 5 that is specific to objects rotating in a circle — θ is the radian measure of the angle (in standard position) where the rotation starts. This interpretation does not make sense in Carlotta's situation, however. Carlotta will need to think about θ in a different way.

Let's start by taking a look at the function from Problem 2 that gives the height y in feet of Terrell's ball above the ground with respect the number of seconds t since he started twirling. This function is defined by the formula

$$y = h(t) = 5 + 2.2\sin\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right)$$

We know that Terrell *starts twirling the ball* at an angle of $\theta = -\frac{2\pi}{3}$ `rad`. Let's think about the function in terms of inputs and outputs. The *input* into the function h will be values of the variable t, and this variable represents the number of seconds since Terrell started twirling the ball. However, we know

The sine function only accepts inputs that are measured in radians.

Consequently, the function

$$\beta = g(t) = \frac{\pi}{3}t - \frac{2\pi}{3}$$

must convert the value of t (measured in seconds) into an angle measure in radians. Of course, we know that it does — the function g gives us the radian measure of the rotation angle in standard position (whose terminal side is the string attached to the ball) t seconds after Terrell started twirling.