MATH 4510 EXAM II
PART II — TAKE HOME EXAM
100 points

NAME:____________________________

This portion of the exam will be due at the beginning of class on Tuesday December 1, 2015. You may not work together or use any outside sources for this portion of the exam. Work all problems neatly on your own paper and attach this coversheet to your exam. By signing this page, you testify that you did not work with anyone or use outside sources to help you. You may use your portfolios.

20 pts 1. Let $\mathcal{G} = (G, *)$ be any group, and let $a \in G$. Prove that the set $H_a = \{a^n : n \in \mathbb{Z}\}$ is a subgroup of $\mathcal{G}$.

20 pts 2. Let $\mathcal{M}_2$ represent the group of all 2×2 invertible matrices under the operation of matrix multiplication. Let

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{Q}^* \right\}$$

be the set of all 2×2 diagonal matrices with nonzero rational entries on the main diagonal. Prove that H is a subgroup of the group $\mathcal{M}_2$.

3. Let $\mathcal{G} = (G, *)$ be any group, and let $a \in G$. An element $b \in G$ is said to commute with a provided $a * b = b * a$.

10 pts (a) If $x \in G$ commutes with a, prove that x^{-1} also commutes with a. (It helps to remember that $a * x^{-1} = e * a * x^{-1}$, where e is the identity element for $\mathcal{G}$.)

10 pts (b) Let C_a represent the set of all elements in G that commute with a. Prove that the set C_a is a subgroup of $\mathcal{G}$.

20 pts 4. Suppose that $\mathcal{G} = (G, *)$ and $\mathcal{H} = (H, \#)$ are groups, and suppose that $f : G \longrightarrow H$ is an isomorphism. If X is a subgroup of $\mathcal{G}$, prove that the set $f(X) = \{f(a) : a \in X\}$ is a subgroup of $\mathcal{H}$.

5. Suppose that $\mathcal{G} = (G, *)$ and $\mathcal{H} = (H, \#)$ are groups and let $\mathcal{G} \times \mathcal{H}$ represent the product group.

10 pts (a) Let e be the identity for the group $\mathcal{H}$. Show that the set $X = \{(x, e) : x \in G\}$ is a subgroup of $\mathcal{G} \times \mathcal{H}$.

10 pts (b) Show that $\mathcal{G}$ is isomorphic to a subgroup of $\mathcal{G} \times \mathcal{H}$.